THE MULTISET PARTITION ALGEBRA: DIAGRAM-LIKE BASES AND REPRESENTATIONS

A Thesis

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Abstract

There is a classical connection between the representation theory of the symmetric group and the general linear group called Schur–Weyl Duality. Variations on this principle yield analogous connections between the symmetric group and other objects such as the partition algebra and more recently the multiset partition algebra. The partition algebra has a well-known basis indexed by graph-theoretic diagrams which allows the multiplication in the algebra to be understood visually as combinations of these diagrams. My thesis begins with a construction of an analogous basis for the multiset partition algebra. It continues with applications of this basis to constructing the irreducible representations of the multiset partition algebra and analysis of subalgebras analogous to the Tanabe algebras and the Brauer algebra. Finally, I address connections between the multiset partition algebra and longstanding questions in the representation theory of the symmetric group including the Kronecker problem and the restriction problem from GL_n to S_n.

Preface

First, I would like to thank my advisor Rosa Orellana. This work could not have been possible without her guidance, encouragement and wisdom. I've learned a great deal from her perspective and she has always been willing to share it.

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Contents

List of Symbols

$\mathbb{C}$ The field of complex numbers

$a.v$ An algebra or group element a acting on a vector v

$\mathrm{Hom}(V, W)$ The homomorphisms between vector spaces V and W

$\mathrm{Hom}_A(V, W)$ The homomorphisms between A-modules V and W

$\mathrm{End}(V)$ The endomorphisms of a vector space

$\mathrm{End}_A(V)$ The endomorphisms of an A-module

$V \oplus W$ The direct sum of two vector spaces or A-modules

$V \otimes W$ The tensor product of two vector spaces or A-modules

$\lambda \in \Lambda^A$ For a semisimple algebra A the indexing set for its simple modules

W_A^λ For a semisimple algebra A a representative of the isomorphism class of simple A-modules indexed by λ

A^λ For a semisimple algebra A a construction of a simple A-module isomorphic to W_A^λ as a combinatorial action

$\mathbb{C}G$ The group ring of G

$\mathbb{C}\{X\}$ The vector space of formal linear combinations of the elements of X

$\mathrm{Res}_B^A(V)$ The restriction of the A-module V to B

$\mathrm{Ind}_B^A(V)$ The induced representation of the B-module V to A

$\sigma \in \mathfrak{S}_n$ The symmetric group on n symbols

$\lambda \vdash n$ λ is a partition of n

$\ell(\lambda)$ the number of parts, or length, of a partition λ

$\mathfrak{S}_\lambda$ or $\mathfrak{S}_{\boldsymbol{a}}$ A Young subgroup of the symmetric group

M^λ The permutation module corresponding to λ

$[n]$ The numbers $\{1, \ldots, n\}$

$t \in \mathcal{YT}(\lambda)$ The set of Young tableaux of shape λ

$\{t\}$ The tabloid corresponding to the Young tableau t

C_t The column-stabilizer of $t \in \mathcal{YT}(\lambda)$

c_t The signed sum of the elements of C_t

n_t The polytabloid corresponding to $t \in \mathcal{YT}(\lambda)$

S^λ The Specht module corresponding to λ

$t \in \mathcal{SYT}(\lambda)$ The set of standard Young tableaux of shape λ

$g_{A,B}$ A Garnir element

$g^{\nu}_{\lambda\mu}$ The Kronecker coefficient

GL_n The general linear group of invertible $n \times n$ matrices over $\mathbb{C}$

V_n An n-dimensional complex vector space

${V_n}^{\otimes r}$ The rth tensor power of V_n

${V_n}^{\otimes \lambda}$ The $|\lambda|$th tensor power of V_n written in Young diagrams

a_λ The Young symmetrizer corresponding to λ

$c^{\nu}_{\lambda\mu}$ The Littlewood-Richardson coefficient

$V_{n,k}$ The space of $n \times k$ matrices over $\mathbb{C}$

$\mathcal{P}^r(V_{n,k})$ The space of homogeneous polynomial forms of degree r on $V_{n,k}$

$P_r(n)$ The partition algebra

$\pi \in \Pi_{2r}$ The set of set partitions of $[r] \cup [\overline{r}]$

$[\overline{n}]$ The barred numbers $\{\overline{1}, \ldots, \overline{n}\}$

$\ell(\pi)$ The number of blocks of a set partition π

$S < R$ For sets S and R, the last-letter order

$\mathcal{T}_\pi$ An element of the orbit basis of $P_r(n)$

$\mathcal{L}_\pi$ An element of the diagram basis of $P_r(n)$

$\pi \leq \nu$ For set partitions π and ν, the coarsening relation

$\mathrm{rank}(\pi)$ The number of propagating blocks of a set partition π

π^T The transpose of π

O_n The orthogonal group

$U_q(\mathfrak{gl}_2)$ The quantum group of the Lie algebra $\mathfrak{gl}_2$

$B_r(n)$ The Brauer algebra

$TL_r(n)$ The Temperley-Lieb algebra

$w \in \mathcal{W}_{P_r}^m$ The set of symmetric m-diagrams

$\sigma_{\pi,w}$ The twist of the conjugation of w by π

$\pi \circ \nu$ The concatenation of π and ν

$\ell(\pi,\nu)$ The number of components entirely in the middle removed in the concatenation of π and ν

$T \in \mathcal{SPT}(\lambda, r)$ The set of set partition tableaux of shape λ and content a set partition of $[r]$

$T \in \mathcal{SSPT}(\lambda, r)$ The set of standard set partition tableaux of shape λ and content a set partition of $[r]$

v_T A basis element of P_r^λ

$\mathbb{MP}_{r,k}(n)$ The multiset partition algebra

$m_s(\tilde{S})$ The multiplicity of the element s in the multiset $\tilde{S}$

$\tilde{S} \uplus \tilde{R}$ The union of two multisets

$\tilde{\pi} \in \tilde{\Pi}_{2r,k}$ The set of multiset partitions with r elements from $[k]$ and r elements from $[\overline{k}]$

$\mathcal{X}_{\tilde{\pi}}$ An element of the orbit basis for $\mathbb{MP}_{r,k}(n)$

$\tilde{T} \in \mathcal{MPT}(\lambda, r, k)$ The set of multiset partition tableaux of shape λ and content a multiset partition of $[k]$ with a total of r elements

$\tilde{T} \in \mathcal{SSMPT}(\lambda, r, k)$ The set of semistandard multiset partition tableaux of shape λ and content a multiset partition of $[k]$ with a total of r elements

$t \leftarrow m$ The row-insertion of the value m into the tableau t

$P(\sigma)$ The insertion tableau of a generalized permutation σ

$Q(\sigma)$ The recording tableau of a generalized permutation σ

$\lambda[n]$ For a partition with $n - |\lambda| \geq \lambda_1$, the partition obtained by adding a long enough first row to create a partition of n

$\biguplus$ Disjoint union

$\tilde{B}$ For an algebra B with distinguished idempotents, the corresponding painted algebra

$\tilde{M}$ For a B-module, the corresponding painted module

$\boldsymbol{a} \in W_{r,k}$ The set of weak compositions of r of length k

$s_{\boldsymbol{a}}$ The idempotent corresponding to the Young subgroup $\mathfrak{S}_{\boldsymbol{a}}$

$\mathcal{L}_\sigma$ The diagram-basis element corresponding to the permutation σ

$\kappa_{\boldsymbol{a},\boldsymbol{b}}$ The coloring map

$U_{\boldsymbol{a}}$ The subspace of $\mathcal{P}^r(V_{n,k})$ spanned by monomials formed by indeterminates whose second label multiplicities are given by $\boldsymbol{a}$

$\mathcal{D}_{\tilde{\pi}}$ An element of the diagram-like basis for $\mathbb{MP}_{r,k}(n)$

$w_{\tilde{T}}$ A basis element of $\mathbb{MP}^{\lambda}_{r,k}$

$N(\tilde{\pi})$ The multiset of nonbasic blocks of a multiset partition $\tilde{\pi}$

$\mathrm{nbw}(\tilde{\pi})$ The nonbasic weight of $\tilde{\pi}$

$\tilde{\pi}/\tilde{B}$ The removal of $\tilde{B}$ from $\tilde{\pi}$

$\mathrm{vb}(\tilde{\pi})$ The number of vertical bars in $\tilde{\pi}$

$\tilde{\pi} \prec \tilde{\tau}$ A partial order defined on multiset partitions just before Lemma 4.3

$\mathbb{MP}_{\boldsymbol{a}}(n)$ The balanced multiset partition algebra

$\tilde{T} \in \mathcal{SSMPT}(\lambda, \boldsymbol{a})$ The set of semistandard multiset partition tableaux of shape λ with $\boldsymbol{a}_i$ i's for each $i \in [k]$

Chapter 1

Preliminaries

Section 1.1

Outline

In this thesis we consider the algebras of endomorphisms of symmetric group modules. The representation theory of these algebras, called centralizer algebras, is closely connected to that of the symmetric group and allows for longstanding questions about symmetric group modules to be rephrased in terms of new combinatorial objects. The work in this thesis is built upon a larger project started by Orellana and Zabrocki who introduced the multiset partition algebra [41] and the character symmetric functions [40] as tools to study the Kronecker problem and restriction problem. We present a nice basis for their algebra whose product admits a combinatorial description on graph-theoretic diagrams and use this basis to construct irreducible representations and a generating set for the algebra. The ultimate goal of this work is to better understand the representation theory of the symmetric group.

The thesis is organized as follows.

Chapter 1 sets the stage by introducing the requisite algebraic objects and telling a story of how combinatorial objects naturally arise to study them. We give a brief review of terminology and theorems in the representation theory of semisimple algebras in Section 1.2.1 and Section 1.2.2. In Section 1.2.3 and Section 1.2.4 we give an overview of representations of the symmetric group and general linear group, and in Section 1.2.5 we give a general duality theorem explaining why tableaux occur naturally to describe the representations of these two objects as well as motivating the study of centralizer algebras. In Section 1.3, we introduce the partition algebra and multiset partition algebra as centralizer algebras of the symmetric group. The partition algebra has a beautiful diagram-theoretic product and a description of its simple modules as actions on set-valued tableaux. The chapter concludes by introducing the RSK algorithm as a combinatorial explanation of dimension formulas appearing in representation theory. This chapter gives no new results, but does attempt to provide a unified presentation of the objects of study.

Chapter 2 is concerned with finding a basis for the multiset partition algebra with a diagram-theoretic multiplication analogous to that of the partition algebra. Section 2.1 begins by introducing a construction called the painted algebra of an algebra with some distinguished idempotents as well as its corresponding painted modules. Section 2.2 examines the orbits of a Young subgroup action on set partitions and tableaux to bridge the gap between these objects built on sets and those built on multisets. Finally, Section 2.3 combines the results of these two earlier sections to construct the diagram-like basis whose product is given by an averaged version of the usual product in the partition algebra.

Chapter 3 moves on to the construction the irreducible representations of the multiset partition algebra as actions on multiset-valued tableaux. We accomplish this by considering projections of simple partition algebra modules via certain idempotents along with a modified RSK algorithm to provide a basis for the modules. The action on these modules is given by an averaged version of the usual action of the partition algebra. The chapter concludes with a formula for computing the character of a diagram-like basis element acting on a simple module.

Chapter 4 provides a generating set for the multiset partition algebra. We accomplish this by providing a sort of factoring that lets a certain class of blocks be removed from a diagram, writing it as a polynomial in simpler (that is, smaller in a particular partial order) diagrams. This gives a recursive algorithm for writing any diagram as a polynomial in the set of generators.

Chapter 5 concludes with a discussion of applications of the objects and methods in the thesis to the study of the representation theory of the symmetric group. It begins by introducing a smaller algebra called the balanced multiset partition algebra $\mathbb{MP}_{\boldsymbol{a}}(n)$ for which the connections to the symmetric group can be more simply stated. Then we introduce the Kronecker problem and show how, similarly to the partition algebra, Kronecker coefficients appear when restricting irreducible $\mathbb{MP}_{\boldsymbol{a}}(n)$-modules to Young-type subalgebras. Then in Section 5.3 we introduce the restriction problem and show how the representation theory of $\mathbb{MP}_{\boldsymbol{a}}(n)$ give a new proof for existing combinatorial interpretations of restriction coefficients of the symmetric power. In Section 5.3.2 we give a combinatorial interpretation for the restriction coefficients of the exterior power by considering an alternating version of the multiset partition algebra. Finally, in Section 5.3.3 we explain how these methods could be extended to study the restriction problem more generally.

Appendix A provides an explicit isomorphism between the generic multiset partition algebra introduced by Orellana and Zabrocki and the painted partition algebra. This isomorphism provides a formula for the change of basis from their orbit basis to the diagram-like basis.

Section 1.2

Representation Theory

We begin this chapter by reviewing some terminology in the representation theory of finite-dimensional algebras and providing some references for a reader who wants to become more familiar with them. We then walk through the representation theory of the symmetric group and general linear group in some detail because the combinatorial objects which describe their simple modules form a model for the representations we later construct in Chapter 3. We conclude with a discussion of Schur–Weyl duality in its various forms.

1.2.1. Representations of Finite Dimensional Algebras

In this thesis, an *algebra* A is finite dimensional over $\mathbb{C}$ and has an identity element 1. An element $e \in A$ is called an *idempotent* if $e^2 = e$.

An A-*module* V is a vector space over $\mathbb{C}$ with a linear action $A \times V \to V$ written $(a, v) \mapsto a.v$. In the literature, this object is also called a *representation*. We will primarily stick with the module terminology and for us a representation will mean a homomorphism $\rho : A \to \mathrm{End}(V)$. These objects are equivalent where the action on V is given by $a.v = \rho(a)(v)$. For A-modules V and W, we write $\mathrm{Hom}_A(V, W)$ for the A-*homomorphisms* from V to W. That is, the set of linear maps $\varphi : V \to W$ such that $a.\varphi(v) = \varphi(a.v)$. We write $\mathrm{End}_A(V) = \mathrm{Hom}_A(V, V)$. We say two A-modules V and W are *isomorphic* if there exists $\varphi \in \mathrm{Hom}_A(V, W)$ which is invertible with $\varphi^{-1} \in \mathrm{Hom}_A(W, V)$.

A *submodule* of an A-module V is a subspace W of V which is A-*invariant*. That is, $a.w \in W$ for each $a \in A$ and $w \in W$. An A-module V is called *simple* if it has no submodules other than $\{0\}$ and V itself. The following characterization of simple modules will be useful to us.

Theorem 1.1 ([13, Proposition 1.1])**.** *The following are equivalent for an A-module V:*

(a) V is simple.

(b) V is generated by any nonzero element (that is, $V = \{a.v : a \in A\}$ for any $v \in V$ nonzero).

(c) $V \cong A/I$ for some maximal left ideal I.

Given two A-modules V and W, their *direct sum* as A-modules is $V \oplus W = \{(v, w) : v \in V, w \in W\}$ where $a.(v, w) = (a.v, a.w)$. We say that V is a *semisimple module* if

$$V = S_1 \oplus \cdots \oplus S_n,$$

where each S_i is simple. We also have a useful characterization of semisimple modules:

Theorem 1.2 ([13, Proposition 1.4, Proposition 1.6])**.** *An A-module V is semisimple if and only if each submodule W is a direct summand. That is, there exists a complementary submodule W' such that $V = W \oplus W'$.*

A particularly important A-module is the (left) *regular representation*. This is the module in which A acts on itself by left-multiplication. If A is a semisimple module over itself in this way, then we say that A is a *semisimple algebra*. The following theorem demonstrates that this is a powerful condition.

Theorem 1.3 ([13, Theorem 1.9])**.** *Let A be an algebra. The following are equivalent:*

(a) A is a semisimple algebra.

(b) Every A-module is semisimple.

(c) Every short exact sequence of A-modules splits.

Moreover, if these conditions hold, then every simple A-module is isomorphic to a submodule of A. In particular, there are only finitely many simple A-modules (up to isomorphism).

This theorem tells us that each A-module can be built up from simple A-modules and that there are only a finite number of these simple building blocks. For a semisimple algebra A, we write Λ^A for the indexing set for the simple A-modules. We write W_A^λ for a representative element of the equivalence class of simple A-modules indexed by λ. In contrast, we will write A^λ for such an irreducible representation which is described as an action on combinatorial objects like tableaux (see Example 1.7 for our first example of such a description). The former perspective will be useful when we state the duality theorems in Section 1.2.5 and the latter allows us to better understand the modules.

It is a consqeuence of the Wedderburn Structure Theorem ([13, Theorem 1.11]) that the regular representation of a semisimple algebra A decomposes into simple A-modules as

$$A \cong \bigoplus_{\lambda \in \Lambda^A} \left(W_A^\lambda\right)^{\oplus m_\lambda}$$

where $m_\lambda = \dim(W_A^\lambda)$. That is, each simple A-module occurs with multiplicity equal to its dimension in the regular representation of A. This decomposition leads to a general statement relating the dimension of the algebra A with the dimension of its simple modules:

$$\dim A = \sum_{\lambda \in \Lambda^A} (\dim W_A^\lambda)^2. \tag{1.1}$$

Of particular interest to us will be the case of a group ring for a finite group G. Given a group G, its *group algebra* $\mathbb{C}G$ has basis G with the group product extended linearly. We will write G-module or G representation to mean a module or representation of the group ring $\mathbb{C}G$.

Theorem 1.4 (Maschke's Theorem)**.** *Let G be a finite group. Then the group ring $\mathbb{C}G$ is semisimple.*

1.2.2. Restriction and Induction

Given a subalgebra B of A, any A-module V can be viewed as a B-module via restriction. We write $\mathrm{Res}_B^A(V)$ for this B-module and call it the *restricted representation*. If V is a simple A-module, it may *not* be the case that $\mathrm{Res}_B^A(V)$ is a simple B-module. In fact, we discuss the difficulty of this problem for the general linear group GL_n and the symmetric group $\mathfrak{S}_n$ in Section 5.3.

We can also go the other way. Given a B-module W, we can define an A-module called the *induced representation* by

$$\mathrm{Ind}_B^A(W) = A \otimes_B W$$

where B acts on A by right-multiplication. To provide a concrete example of this phenomenon and to set the stage for the following section, we describe how the trivial module of a group can be induced to a larger group (see [45, Section 1.12] for more details on the induced representations of a group).

Let H be a subgroup of G and let $W = \mathbb{C}\{w\}$ be the H-module with action $h.w = w$ for all $h \in H$. This module W is called the *trivial module* of H. Notice that $g \otimes_{\mathbb{C}H} w = g' \otimes_{\mathbb{C}H} w$ if and only if $g = g'h$ for some $h \in H$. That is, the corresponding cosets $gH = g'H$ are identical. Hence, if $\{g_1H, \ldots, g_nH\}$ is a complete set of cosets of H in G, then

$$\mathrm{Ind}_H^G(W) \cong \mathbb{C}\{g_1H, \ldots, g_nH\}$$

via the map $g \otimes_{\mathbb{C}H} w \mapsto gH$ where the action of G on the cosets is left-multiplication.

1.2.3. Representations of the Symmetric Group

In this section, we specialize to the case of the symmetric group because the combinatorial description of its modules serves as a model for the representations we construct. Our treatment closely follows [45].

From elementary representation theory, we know that the number of irreducible representations of $\mathfrak{S}_n$ is equal to the number of conjugacy classes of $\mathfrak{S}_n$. The conjugacy class that $\sigma \in \mathfrak{S}_n$ belongs to is determined by its cycle structure. To track this cycle structure, we write the sizes of σ's cycles in weakly decreasing order. Such a weakly decreasing sequence λ of positive integers summing to n is called a *partition* of n. We write $\lambda \vdash n$ to say that λ is a partition of n and $\ell(\lambda)$ for the length (the number of integers, called parts, in the partition).

One way that we could try to construct a representation for each $\lambda \vdash n$ would be to start with a subgroup corresponding to λ and induce the trivial representation up to $\mathfrak{S}_n$. For $\lambda \vdash n$, write

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times \mathfrak{S}_{\{n-\lambda_\ell+1,\ldots,n\}}$$

for the *Young subgroup* corresponding to λ. For each $\lambda \vdash n$, let W be the trivial $\mathfrak{S}_\lambda$-module and define the $\mathfrak{S}_n$-module

$$M^\lambda = \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(W) = \mathbb{C}\{\sigma\mathfrak{S}_\lambda : \sigma \in \mathfrak{S}_n\}$$

called the *permutation module* corresponding to λ. It is far too much to hope that these modules are irreducible, but they will naturally lead us to the objects we need to construct the irreducible modules of $\mathfrak{S}_n$. For these objects, it will be useful to write a permutation $\sigma \in \mathfrak{S}_n$ in *one-line notation*. That is, σ is written as

$$\sigma(1)\sigma(2)\ldots\sigma(n).$$

Example 1.5. As an example, we analyze the coset $21345\mathfrak{S}_{(2,2,1)}$ in $M^{(2,2,1)}$.

$$21345\mathfrak{S}_{(2,2,1)} = \{21345, 12345, 21435, 12435\}$$

If we represent each permutation as filling an array of boxes with row-lengths given by the partition $(2,2,1)$, we see a pattern emerge:

$$\begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array}\,\mathfrak{S}_{(2,2,1)} = \left\{ \begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array}\,, \begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}\,, \begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array}\,, \begin{array}{|c|c|}\cline{1-1} 5 & \multicolumn{1}{c}{} \\ \hline 4 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \right\}$$

Notice that the permutations in the cosets are precisely those obtained by rearranging numbers in the same row of these arrays.

Inspired by the above example, we define these arrays formally. For $\lambda \vdash n$, the *Young diagram* of λ is a left-justified array of boxes with λ_i boxes in the ith row from the bottom. A *Young tableau* t of shape λ is a filling of the boxes of λ's Young diagram with the numbers $[n] = \{1, \ldots, n\}$. Write $\mathcal{YT}(\lambda)$ for the set of these tableaux. There are $n!$ Young tableaux of shape $\lambda \vdash n$. A permutation $\sigma = \sigma(1) \ldots \sigma(n)$ in one-line notation can be taken to its corresponding tableau by filling the diagram with entries $\sigma(1), \sigma(2), \ldots, \sigma(n)$ left-to-right starting with the bottom row. We want to consider what happens to a permutation when it is multiplied on the right by the Young subgroup $\mathfrak{S}_\lambda$. Note that right multiplication by a permutation τ will rearrange the entries in the one-line notation and that the permutations in $\mathfrak{S}_\lambda$ are precisely the permutations that only rearrange entries of the permutation that are in the same row of the corresponding tableau. Hence, the cosets $\sigma\mathfrak{S}_\lambda$ correspond to Young tableaux of shape λ up to rearranging the entries in each row. We define the *tabloid* $\{t\}$ of a Young tableau t to be the equivalence class of Young tableaux with the same set of numbers in each row as t. We write tabloids as arrays of integers with only horizontal lines separating each row to indicate that the numbers are free to be re-ordered within rows.

Example 1.6. The tabloid corresponding to the coset $21345\mathfrak{S}_{(2,2,1)}$:

$$21345\mathfrak{S}_{(2,2,1)} = \left\{ \begin{array}{|c|c|}\cline{1-1} 5 \\ \hline 3 & 4 \\ \hline 2 & 1 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\cline{1-1} 5 \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\cline{1-1} 5 \\ \hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\cline{1-1} 5 \\ \hline 4 & 3 \\ \hline 1 & 2 \\ \hline \end{array} \right\} = \begin{array}{cc}\cline{1-1} 5 \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$$

Now we consider the action of $\mathfrak{S}_n$ on M^λ by left-multiplication. In contrast to right-multiplication, multiplying a permutation by some $\sigma \in \mathfrak{S}_n$ on the left replaces each i in its one-line notation with $\sigma(i)$. The action of $\sigma \in \mathfrak{S}_n$ on a tabloid $\{t\}$ then replaces each entry i of $\{t\}$ with $\sigma(i)$.

Example 1.7. The action of $\sigma = 32154$ on the tabloid above is

$$\begin{aligned} \sigma.\ \begin{array}{cc}\cline{1-1} 5 \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} &= \begin{array}{cc}\cline{1-1} \sigma(5) \\ \hline \sigma(3) & \sigma(4) \\ \hline \sigma(1) & \sigma(2) \\ \hline \end{array} \\ &= \begin{array}{cc}\cline{1-1} 4 \\ \hline 1 & 5 \\ \hline 3 & 2 \\ \hline \end{array}\ . \end{aligned}$$

This action is our first example of constructing a representation as a combinatorial action, but recall that our goal is to construct the *irreducible* representations of $\mathfrak{S}_n$. To this end, we want to associate an element of M^λ to each tableau $t \in \mathcal{YT}(\lambda)$. For $t \in \mathcal{YT}(\lambda)$ with columns $C_1, \ldots, C_k$, let $C_t = \mathfrak{S}_{C_1} \times \cdots \times \mathfrak{S}_{C_k}$ and

$$c_t = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\sigma.$$

Define the *polytabloid* of t to be

$$n_t = c_t\{t\} \in M^\lambda.$$

Example 1.8. For the tableau

$$t = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}\ ,$$

the corresponding polytabloid is:

$$n_t = \begin{array}{cc}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} - \begin{array}{cc}\hline 1 & 4 \\ \hline 3 & 2 \\ \hline \end{array} - \begin{array}{cc}\hline 3 & 2 \\ \hline 1 & 4 \\ \hline \end{array} + \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}\ .$$

The action of $\sigma \in \mathfrak{S}_n$ on n_t is just as straightforward as for the tabloids:

$$\begin{aligned} \sigma.n_t &= \sigma c_t\{t\} \\ &= c_{\sigma.t}\sigma\{t\} \\ &= n_{\sigma.t} \end{aligned}$$

where $\sigma.t$ is the result of replacing each i in t with $\sigma(i)$.

Let $S^\lambda = \mathbb{C}\{n_t : t \in \mathcal{YT}(\lambda)\}$ be the free $\mathbb{C}$-vector space of formal linear combinations of the vectors n_t with the above action. We call these the *Specht modules*, and the following theorem is a consequence of the submodule theorem of James [25].

Theorem 1.9 ([45, Theorem 2.4.4] following [25])**.** *The Specht modules $\{S^\lambda : \lambda \vdash n\}$ form a complete set of simple $\mathfrak{S}_n$-modules. That is, they are simple and pairwise nonisomorphic, and any simple $\mathfrak{S}_n$-module is isomorphic to S^μ for some μ.*

While we've defined each Specht module as the span of polytabloids n_t for $t \in \mathcal{YT}(\lambda)$, these are linearly dependent. To give a basis for the Specht modules, we define a *standard Young tableau* as a Young tableau whose entries increase along rows left-to-right and along columns bottom-to-top. Write $\mathcal{SYT}(\lambda)$ for the set of standard

Young tableaux.

Theorem 1.10 ([45, Theorem 2.5.2])**.** *The set $\{n_t : t \in \mathcal{SYT}(\lambda)\}$ forms a basis for the Specht module S^λ.*

The action we described above may result in a non-standard Young tableau, so we need a method called a *straightening algorithm* to write it as a linear combination of standard Young tableaux. First, notice that if $\tau \in C_t$, then

$$\begin{aligned} \tau c_t &= \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\tau\sigma \\ &= \sum_{\tau^{-1}\sigma \in C_t} \operatorname{sgn}(\tau^{-1}\sigma)\tau\tau^{-1}\sigma \\ &= \operatorname{sgn}(\tau)c_t, \end{aligned}$$

and so if we act on n_t by τ we get

$$\begin{aligned} \tau.n_t &= \tau c_t\{t\} \\ &= \operatorname{sgn}(\tau)c_t\{t\} \\ &= \operatorname{sgn}(\tau)n_t. \end{aligned}$$

Hence, by rearranging within a column we only introduce a possible sign and can thereby assume that the columns of our tableau t increase. Now the only obstruction we can encounter to our tableau being standard is a decrease within a row of t. That is, i appears before j in the same row and $i > j$. To handle this situation, we define special elements of $\mathbb{C}\mathfrak{S}_n$ called *Garnir elements*. Let the set A contain i and each entry above it in the same column and let the set B contain j and each entry beneath it in the same column. The Garnir element $g_{A,B}$ is

$$g_{A,B} = \sum_{\sigma} \operatorname{sgn}(\sigma)\sigma$$

where the sum is over permutations σ of $A \cup B$ which keep the sets A and B in increasing order from bottom to top in the tableau.

Example 1.11. We compute the Garnir element for the highlighted subsets of the following tableau:

$$t = \begin{array}{|c|c|c|}\cline{1-2} 8 & 9 & \multicolumn{1}{c}{} \\ \hline 4 & 3 & 6 \\ \hline 1 & 2 & 5 \\ \hline\end{array}$$

$$\begin{aligned} A &= \{4, 8\} \\ B &= \{2, 3\} \\ g_{A,B} &= \epsilon - (3\,4) + (2\,3\,4) + (3\,8\,4) - (2\,3\,8\,4) + (2\,4)(3\,8) \end{aligned}$$

Now we see the result of applying this Garnir element to the tableau:

$$\begin{aligned} g_{A,B}\ \begin{array}{|c|c|c|}\cline{1-2} 8 & 9 & \multicolumn{1}{c}{} \\ \hline 4 & 3 & 6 \\ \hline 1 & 2 & 5 \\ \hline\end{array} &= \begin{array}{|c|c|c|}\cline{1-2} 8 & 9 & \multicolumn{1}{c}{} \\ \hline 4 & 3 & 6 \\ \hline 1 & 2 & 5 \\ \hline\end{array} - \begin{array}{|c|c|c|}\cline{1-2} 8 & 9 & \multicolumn{1}{c}{} \\ \hline 3 & 4 & 6 \\ \hline 1 & 2 & 5 \\ \hline\end{array} + \begin{array}{|c|c|c|}\cline{1-2} 8 & 9 & \multicolumn{1}{c}{} \\ \hline 2 & 4 & 6 \\ \hline 1 & 3 & 5 \\ \hline\end{array} \\ &+ \begin{array}{|c|c|c|}\cline{1-2} 4 & 9 & \multicolumn{1}{c}{} \\ \hline 3 & 8 & 6 \\ \hline 1 & 2 & 5 \\ \hline\end{array} - \begin{array}{|c|c|c|}\cline{1-2} 4 & 9 & \multicolumn{1}{c}{} \\ \hline 2 & 8 & 6 \\ \hline 1 & 3 & 5 \\ \hline\end{array} + \begin{array}{|c|c|c|}\cline{1-2} 3 & 9 & \multicolumn{1}{c}{} \\ \hline 2 & 8 & 6 \\ \hline 1 & 4 & 5 \\ \hline\end{array} \end{aligned} \tag{1.2}$$

On the other hand, if we apply the Garnir element to the corresponding element

v_t in $S^{(3,3,2)}$, we find that

$$g_{A,B}n_t = n_t - n_{t_2} + n_{t_3} + n_{t_4} - n_{t_5} + n_{t_6} = 0$$

where $t_2, \ldots, t_6$ are the second through sixth tableaux on the right-hand side of Equation (1.2). Note that this allows us to solve for v_t in terms of tableaux that do not have an decrease in position we chose, but we sometimes create new decreases in other positions (see $8 > 6$ in the last three tableaux in this example).

Although new decreases may appear in this process, the following theorem explains that if we continue to apply these Garnir elements, we will eventually end up with all standard tableaux.

Theorem 1.12 ([45, Proposition 2.6.3, Theorem 2.6.4] following [42])**.** *If $g_{A,B}$ is a Garnir element for a nonstandard tableau t, then*

$$g_{A,B}n_t = 0.$$

We can then solve for n_t as

$$n_t = -\sum_{\sigma \neq \epsilon} (\operatorname{sgn} \sigma) n_{\sigma.t}.$$

Furthermore, if $\sigma \neq \epsilon$ appears in the element $g_{A,B}$, then $\sigma.t$ is larger in some partial order than t. Hence, this process will eventually terminate, writing n_t as a linear combination of standard tableaux.

We end this section with an open problem in the representation theory of the symmetric group. Given two simple $\mathfrak{S}_n$-modules S^λ and S^μ, we can form a new $\mathfrak{S}_n$-

module $S^\lambda \otimes S^\mu$ where $\sigma \in \mathfrak{S}_n$ acts by $\sigma.(n_t \otimes n_s) = (\sigma.n_t) \otimes (\sigma.n_s)$. This module then decomposes into simple modules as

$$S^\lambda \otimes S^\mu \cong \bigoplus_{\nu \vdash n} (S^\nu)^{\oplus g^\nu_{\lambda\mu}}.$$

The multiplicity $g^\nu_{\lambda\mu}$ is called a *Kronecker coefficient* and the Kronecker problem asks for a combinatorial interpretation of $g^\nu_{\lambda\mu}$. We will discuss applications of centralizer algebras to this problem in Section 5.2.

We recommend Sagan's The Symmetric Group [45, Chapter 2], Fulton and Harris's Representation Theory: A First Course [15, Chapter 4], and Fulton's Young Tableaux [14, Chapter 7] for detailed exposition of the construction of Specht modules.

1.2.4. Polynomial Representations of GL_n

Let GL_n be the group of $n \times n$ invertible matrices over $\mathbb{C}$ called the *general linear group*. In this section we discuss some of the representation theory of GL_n to highlight some similarities with the representation theory of the symmetric group and to provide background for the problems discussed in Chapter 5. A *polynomial representation* V of GL_n is a representation $\rho : GL_n \to \mathrm{End}(V)$ such that in any basis of V, the entries of the matrices $\rho(X)$ are polynomials in the entries of X for each $X \in GL_n$. The simple polynomial representations of GL_n can be constructed as actions on tableaux similarly to $\mathfrak{S}_n$.

For $n > 0$, write $V_n = \mathbb{C}^n$ and $\{e_1, \ldots, e_n\}$ for the standard basis of V_n. The vector space V_n has a natural action of GL_n by matrix-vector multiplication. Now for $r > 0$, write ${V_n}^{\otimes r} = \underbrace{V_n \otimes \cdots \otimes V_n}_{r \text{ times}}$ for the rth tensor power of V_n. The tensor power has a basis $\{e_{\boldsymbol{i}} : \boldsymbol{i} \in [n]^r\}$ indexed by sequences of length r with entries in $[n]$ where

$e_{\boldsymbol{i}} = e_{\boldsymbol{i}_1} \otimes \cdots \otimes e_{\boldsymbol{i}_r}$. The action of GL_n on V_n extends diagonally to the tensor power where $M \in GL_n$ acts by

$$M.(e_{\boldsymbol{i}_1} \otimes \cdots \otimes e_{\boldsymbol{i}_r}) = (Me_{\boldsymbol{i}_1}) \otimes \cdots \otimes (Me_{\boldsymbol{i}_r}).$$

We attempt to mirror the previous section (although the motivation will be less apparent until the following section) by constructing representations of GL_n indexed by partitions $\lambda \vdash r$.

For $\lambda \vdash r$, we write ${V_n}^{\otimes\lambda} \cong {V_n}^{\otimes r}$ for the rth tensor power with its basis elements written in a particular way: filling in the Young diagram of λ.

Example 1.13. An element in ${V_4}^{\otimes(3,2,1,1)}$:

$$e_{(1,1,2,1,4,2,3)} \leftrightarrow \begin{ytableau} e_3 \\ e_2 \\ e_1 & e_4 \\ e_1 & e_1 & e_2 \end{ytableau}$$

We've now constructed a representation of GL_n which acts on these Young diagrams filled with vectors.

Example 1.14. The action of a matrix in GL_2 on an element of ${V_2}^{\otimes(2,1)}$:

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} . \begin{ytableau} e_2 \\ e_1 & e_1 \end{ytableau} = \begin{ytableau} 3e_1 + e_2 \\ 2e_1 & 2e_1 \end{ytableau}$$

$$= 12 \begin{ytableau} e_1 \\ e_1 & e_1 \end{ytableau} + 4 \begin{ytableau} e_2 \\ e_1 & e_1 \end{ytableau}$$

These are certainly not irreducible representations (and they're not even distinct—two partitions of the same number give isomorphic representations). As before, we need to project by some element to obtain the simple modules. Consider the action of the symmetric group $\mathfrak{S}_r$ on ${V_n}^{\otimes r}$ which rearranges the tensor factors by

$$\sigma.(v_1 \otimes \cdots \otimes v_r) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}.$$

Note that this action commutes with the diagonal action of GL_n. Let $t \in \mathcal{SYT}(\lambda)$ be the standard Young tableau obtained by numbering the Young diagram from left-to-right, bottom-to-top. This tableau labels the boxes of the Young diagram so that this action of $\mathfrak{S}_r$ extends to $V^{\otimes \lambda}$. Write $r_t = \sum_{\sigma \in \mathfrak{S}_\lambda} \sigma$ which sums over permutations within rows of t and recall the element c_t from the previous section which sums over permutations within columns of t with a sign. The *Young symmetrizer* is the element $a_\lambda = c_t r_t \in \mathbb{C}\mathfrak{S}_r$. We define

$$GL_n^\lambda := a_\lambda V^{\otimes \lambda}.$$

Because the action of a_λ commutes with the diagonal action of GL_n, this is a GL_n-module. Applying the element a_λ results in a similar straightening algorithm: in particular swapping two elements within a column will change the sign. This means that a tableau with a repeated entry within a column is sent to zero, and we can arrange the indices of our basis elements to be weakly increasing along rows and strictly increasing up columns. This leads to the definition of a *semistandard Young tableau* of shape λ: a filling of the Young diagram of λ with positive integers that increase weakly along rows and strictly up columns. Repeating the above example for $GL_2^{(2,1)}$ as an action on semistandard Young tableaux, we get

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \cdot \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} = 12 \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} + 4 \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}$$

$$= 4 \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array} .$$

Notice that GL_n^λ is nonzero if and only if the Young diagram of λ has at most n rows (that is, λ has at most n parts).

Theorem 1.15 ([14, Theorem 8.2.2])**.** *The GL_n-modules*

$$\{GL_n^\lambda : \ell(\lambda) \le n\}$$

form a complete set of irreducible polynomial representations of GL_n.

To contrast with the Kronecker problem of last section, the decompositions of tensor products of GL_n-modules are well-understood. We write the decomposition as

$$GL^\lambda \otimes GL^\mu \cong \bigoplus_\nu (GL^\nu)^{\oplus c^\nu_{\lambda\mu}}$$

where the multiplicity $c^\nu_{\lambda\mu}$ is called the *Littlewood-Richardson* coefficient. A full description of the rule for computing $c^\nu_{\lambda\mu}$ would take us too far afield, but it is the number of a certain class of semistandard Young tableau of a skew shape (a partition shape with a smaller partition shape taken out of its bottom left corner). The rule was first given in [30] but complete proofs were given by Thomas [51, 52] and Schützenberger [48]. See also [45, Section 4.9].

We recommend Fulton's Young Tableaux [14, Chapter 8] for detailed exposition of the constructions of these simple polynomial representations of GL_n as well as representations beyond just the polynomial ones.

1.2.5. Dualities

The fact that the simple $\mathfrak{S}_n$-modules and the simple polynomial GL_n-modules both have descriptions as actions on tableaux is no coincidence—it is the result of a duality between the two objects. These dualities are the reason the diagram algebras discussed in Section 1.3 have deep connections to the representation theory of the symmetric group and general linear group. In this section, we first introduce two particular instances of these dualities: Schur–Weyl duality and Howe duality. Then, we give the double centralizer theorem, which serves as a general description of these dualities.

Classical Schur–Weyl Duality. We saw that the rth tensor power ${V_n}^{\otimes r}$ has commuting actions of GL_n and $\mathfrak{S}_r$. What is interesting is that an even stronger statement is true: these actions are mutual centralizers. That is, the $\mathfrak{S}_r$ action generates all the maps $\mathrm{End}_{GL_n}({V_n}^{\otimes r})$ and the GL_n-action generates all the maps $\mathrm{End}_{\mathfrak{S}_r}({V_n}^{\otimes r})$. In this way, either action can be completely recovered by asking what maps commute with the other. This phenomenon is the classical Schur–Weyl duality discovered by Schur [47] and popularized by Weyl when he used it to classify the representations of the classical groups such as GL_n, O_n and U_n [55]. An important consequence of the duality is that the decomposition of ${V_n}^{\otimes r}$ as a $\mathfrak{S}_r \times GL_n$-module is given by

$$ {V_n}^{\otimes r} = \bigoplus_{\lambda} S^\lambda \otimes GL_n^\lambda $$

where the direct sum is over partitions $\lambda \vdash r$ with at most n parts.

Like the regular representation's decomposition gave us information relating the dimension of an algebra to the dimensions of its simple modules, this decomposition gives us information about the representation theory of the two objects. For example, we have a one-to-one correspondence between a set of simple modules for $\mathfrak{S}_r$ and GL_n. This means that a subset of simple GL_n modules can be naturally indexed by partitions, which helps explain the usefulness of tableaux in constructing its simple polynomial modules. Additionally, this decomposition relates the dimensions and multiplicities of the simple modules. In particular, the multiplicity of S^λ in $V_n^{\otimes r}$ is the dimension of the representation GL_n^λ (i.e. the number of semistandard Young tableaux of shape λ and largest entry n).

Howe Duality. An analogous duality has to do with an action on polynomial forms on matrices. Let $V_{n,k}$ be the space of $n \times k$ matrices over $\mathbb{C}$. There are natural commuting actions of GL_n (by left-multiplication) and GL_k (by right-multiplication) on $V_{n,k}$. Write $\mathcal{P}^r(V_{n,k})$ for the space of homogeneous polynomial forms of degree r on $V_{n,k}$. These are the maps $V_{n,k} \to \mathbb{C}$ which can be described as homogeneous polynomials in the indeterminates x_{ij} with $1 \leq i \leq n$ and $1 \leq j \leq k$ where the indeterminate x_{ij} picks out the entry in the ith row and jth column.

Example 1.16. An element of $\mathcal{P}^3(V_{2,2})$ applied to a 2×2 matrix:

$$(x_{11}^2 x_{12} - x_{11}x_{21}x_{22})\left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right) = 2^2 \cdot 3 - 2 \cdot 4 \cdot 5 = -28$$

The actions of GL_n and GL_k extend naturally to these polynomial forms as follows:

$$(M.f)(X) = f(M^{-1}X) \qquad \text{for } M \in GL_n$$
$$(N.f)(X) = f(XN) \qquad \text{for } N \in GL_k$$

Roger Howe [24] determined that the actions of GL_n and GL_k on $\mathcal{P}^r(V_{n,k})$ form a mutually centralizing pair much like that found in Schur–Weyl duality. This phenomenon is often called Howe duality in his honor. This duality similarly leads to a decomposition of $\mathcal{P}^r(V_{n,k})$ as a $GL_n \times GL_k$-module as

$$\mathcal{P}^r(V_{n,k}) = \bigoplus_{\lambda} GL_n^{\lambda} \times GL_k^{\lambda}$$

where the direct sum is over partitions $\lambda \vdash r$ with at most $\min(n, k)$ parts.

Double Centralizer Theorem. The fact that these mutually centralizing pairs give rise to useful decompositions that set up a correspondence between irreducible representations of the two groups or algebras is summarized in the following theorem called the *double centralizer theorem*.

Theorem 1.17 ([43, Section 6.2.5][16, Section 4.2.1])**.** *Let A be a semisimple algebra acting faithfully on a module V and set $B = \mathrm{End}_A(V)$. Then B is semisimple and $\mathrm{End}_B(V) \cong A$. Furthermore, there is a set P (a subset of the indexing set of the irreducible representations of A) such that for each $x \in P$, there is an irreducible A-module W_A^x occurring in the decomposition of V as an A-module. If we set $W_B^x = \mathrm{Hom}_A(W_A^x, V)$, then W_B^x is an irreducible B-module and the decomposition of V as* *an $A \otimes B$-module is*

$$V \cong \bigoplus_{x \in P} W_A^x \otimes W_B^x.$$

Moreover, the dimension of W_A^x is equal to the multiplicity of W_B^x in V as a B-module and the dimension of W_B^x is equal to the multiplicity of W_A^x in V as an A-module.

One important consequence is that if we start with a semisimple algebra (for example, the group ring of a finite group), we know that the centralizer will also be semisimple and the two actions will be mutual centralizers. In the second half of the preliminary chapter we focus on some algebras that arise from this scenario as centralizers of the symmetric group.

We recommend Procesi's Lie Groups [43] and Goodman and Wallach's Symmetry, Representations, and Invariants [16] for comprehensive treatments of these dualities.

Section 1.3

Diagram Algebras

In the previous section we introduced classical Schur–Weyl duality between the general linear group GL_n and the symmetric group $\mathfrak{S}_r$ acting on ${V_n}^{\otimes r}$. This duality provides connections between the representation theory of the two groups, so we would like to study other centralizers of symmetric group actions to try and learn more about its representations. In this section, we discuss what happens when the actions of GL_n in Schur–Weyl duality and Howe duality are restricted to the $n \times n$ permutation matrices. The algebras that result have bases naturally indexed by combinatorial objects called set and multiset partitions, which have beautiful diagrammatic representations that help understand the structure of the algebras.

1.3.1. The Partition Algebra

First, we consider the restriction of the GL_n action in classical Schur–Weyl duality to the subgroup of $n \times n$ permutation matrices, which is isomorphic to $\mathfrak{S}_n$. In this section, we lay out the story of this centralizer $P_r(n) = \mathrm{End}_{\mathfrak{S}_n}({V_n}^{\otimes r})$ for $n \geq 2r$ called the *partition algebra*.

The partition algebra was first introduced as a generalization of the Temperley-Lieb algebra and the Potts model in statistical mechanics by Jones [27] and Martin [32, 33]. They gave two bases for the algebra: the orbit basis coming naturally from the centralizer algebra structure and the diagram basis whose product is described by graph-theoretic diagrams. Later, presentations of $P_r(n)$ by generators and relations were given [11, 12, 21] and the dimensions of its irreducible representations were found [4, 5, 40]. From these dimensions, it was predicted that the representations of $P_r(n)$ could be constructed as combinatorial actions on set-valued tableaux. Finally, the irreducible representations were constructed as actions on these tableaux [19, 20].

The Orbit Basis. The algebra $P_r(n)$ has a natural basis which comes from analyzing orbits under an $\mathfrak{S}_n$ action. To see concretely how this basis arises, we first consider what the $\mathfrak{S}_n$ action on ${V_n}^{\otimes r}$ looks like on the basis $\{e_{\boldsymbol{i}} : \boldsymbol{i} \in [n]^r\}$. Because this $\mathfrak{S}_n$ action comes from restricting the diagonal action of GL_n, we have

$$\begin{aligned} \sigma.(e_{i_1} \otimes \cdots \otimes e_{i_n}) &= (\sigma.e_{i_1}) \otimes \cdots \otimes (\sigma.e_{i_n}) \\ &= e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_n)}. \end{aligned}$$

To simplify the computations to come, we introduce the notation

$$\sigma(\boldsymbol{i}) = (\sigma(i_1), \sigma(i_2), \cdots \sigma(i_n))$$

so that the above expression can be written $\sigma.e_{\boldsymbol{i}} = e_{\sigma(\boldsymbol{i})}$.

The action of $\mathfrak{S}_n$ on ${V_n}^{\otimes r}$ extends naturally to $\mathrm{End}({V_n}^{\otimes r})$ by conjugation. That is, for $f \in \mathrm{End}({V_n}^{\otimes r})$ and $\sigma \in \mathfrak{S}_n$,

$$(\sigma.f)(v) = \sigma.f(\sigma^{-1}.v).$$

Notice that $f \in \mathrm{End}_{\mathfrak{S}_n}({V_n}^{\otimes r})$ if and only if $\sigma.f = f$.

Our basis of ${V_n}^{\otimes r}$ gives rise to the basis $\{E^{\boldsymbol{i}}_{\boldsymbol{j}} : (\boldsymbol{i}, \boldsymbol{j}) \in [n]^r \times [n]^r\}$ for $\mathrm{End}({V_n}^{\otimes r})$ where

$$E^{\boldsymbol{i}}_{\boldsymbol{j}} e_{\boldsymbol{k}} = \delta_{\boldsymbol{i},\boldsymbol{k}} e_{\boldsymbol{j}}.$$

We can then describe the $\mathfrak{S}_n$-action on $\mathrm{End}({V_n}^{\otimes r})$ with respect to this basis by

$$\begin{aligned}
\sigma.E^{\boldsymbol{i}}_{\boldsymbol{j}} e_{\boldsymbol{k}} &= \sigma.E^{\boldsymbol{i}}_{\boldsymbol{j}} \sigma^{-1}.e_{\boldsymbol{k}} \\
&= \delta_{i,\sigma^{-1}(\boldsymbol{k})} e_{\sigma(\boldsymbol{j})} \\
&= \delta_{\sigma(i),\boldsymbol{k}} e_{\sigma(\boldsymbol{j})} \\
&= E^{\sigma(\boldsymbol{i})}_{\sigma(\boldsymbol{j})} e_{\boldsymbol{k}},
\end{aligned}$$

so $\sigma.E^{\boldsymbol{i}}_{\boldsymbol{j}} = E^{\sigma(\boldsymbol{i})}_{\sigma(\boldsymbol{j})}$.

Consider the sum

$$\sum_{(\boldsymbol{i}',\boldsymbol{j}') \in \mathfrak{S}_n.(\boldsymbol{i},\boldsymbol{j})} E^{\boldsymbol{i}'}_{\boldsymbol{j}'}.$$

over the orbit of a basis element $E^{\boldsymbol{i}}_{\boldsymbol{j}}$. The collection of these sums for each orbit form a basis of $\mathrm{End}_{\mathfrak{S}_n}(V_n^{\otimes r})$. We would then like to have a nice way to represent these orbits so we can provide an indexing set for this basis of $P_r(n)$. As an example, we look at the orbit of $(\boldsymbol{i}, \boldsymbol{j}) = (213, 233)$ under the $\mathfrak{S}_3$-action given in the table below.

	$\boldsymbol{i}$	$\boldsymbol{j}$
ϵ	213	233
(12)	123	133
(13)	231	211
(23)	312	322
(123)	321	311
(132)	132	122

The data of this table can be condensed to the single pair (abc, acc) where each row can be recovered by assigning a, b, and c to different values in $\{1, 2, 3\}$. In fact, this is necessarily the case for the orbit of any pair of sequences in $[n]^r \times [n]^r$ under the $\mathfrak{S}_n$-action. Because each σ is an injective map, any two values among the sequences which were the same (resp. different) before applying σ remain the same (resp. different) after. The pair (abc, acc) can further be associated to an object called a set partition.

A *set partition* π of a set S is a set of $\ell(\pi)$ nonempty subsets of S called *blocks* whose disjoint union is S. To convert the above orbit into a set partition, we index the elements of the first sequence with barred numbers $[\overline{3}] = \{\overline{1}, \overline{2}, \overline{3}\}$ and the elements of the second sequence with $[3] = \{1, 2, 3\}$. We put two numbers from $[3] \cup [\overline{3}]$ in the same block if and only if they correspond to the same letter.

$$\left(\tfrac{abc}{123}, \tfrac{acc}{123}\right) \longleftrightarrow \{\{1, \overline{1}\}, \{\overline{2}\}, \{2, 3, \overline{3}\}\}$$

The blocks within a set partition do not have a set order, so we would like a standard order to write the blocks in called the *last letter order*. For blocks S and R, we say that $S < R$ if one of the following conditions hold:

(a) S is empty and R is not.

(b) $\max(S) < \max(R)$.

(c) $\max(S) = \max(R) = m$ and $S \setminus \{m\} < R \setminus \{m\}$.

The blocks in the above example are written in last-letter order where all barred numbers are considered larger then all unbarred numbers.

In this way, the orbits of the $\mathfrak{S}_n$-action on $\{E_{\boldsymbol{j}}^{\boldsymbol{i}} : (\boldsymbol{i}, \boldsymbol{j}) \in [n]^r \times [n]^r\}$ correspond to set partitions of $[r] \cup [\overline{r}]$ with at most n blocks. Because we assume that $n \geq 2r$, we obtain all set partitions of $[r] \cup [\overline{r}]$. Write Π_{2r} for the set of these set partitions. We write

$$\mathcal{T}_\pi = \sum_{\sigma \in \mathfrak{S}_n} E_{\sigma(\boldsymbol{j})}^{\sigma(\boldsymbol{i})}$$

for the corresponding basis element and call the basis $\{\mathcal{T}_\pi : \pi \in \Pi_{2r}\}$ the *orbit basis*.

We will delay providing the product formula for the orbit basis until it is needed in Appendix A, but we provide the following example. The reader should compare this example to the product given in the following section to understand the necessity of changing to a different basis.

Example 1.18. The product of two orbit-basis elements:

$$\mathcal{T}_{12/\bar{1}/3\bar{2}\bar{3}}\mathcal{T}_{1/\bar{1}/\bar{2}/23\bar{3}} = (n-4)\mathcal{T}_{12/\bar{1}/\bar{2}/3\bar{3}} + (n-3)\mathcal{T}_{12\bar{1}/\bar{2}/3\bar{3}} + (n-3)\mathcal{T}_{12\bar{2}/\bar{1}/3\bar{3}} + (n-2)\mathcal{T}_{12\bar{1}\bar{2}/3\bar{3}}$$

Partition Diagrams and The Diagram Basis. There is a second basis called the *diagram basis* of $P_r(n)$ whose product can be described on graph-theoretic diagrams. Properties of these diagrams make the product easy to perform and explain some algebraic properties of $P_r(n)$.

If $\pi = \{S_1, \ldots, S_k\}$ and $\nu = \{R_1, \ldots, R_\ell\}$ are set partitions, we say that $\pi \leq \nu$, or π is *coarser* than ν, if π can be obtained by combining blocks of ν. Equivalently, each block S_i of π is contained in some block R_j of ν. For example,

$$\{\{1,2,4,5\},\{3,6\}\} \leq \{\{1,4\},\{2,5\},\{3\},\{6\}\}.$$

We define a second basis $\{\mathcal{L}_\pi : \pi \in \Pi_{2r}\}$ for $P_r(n)$ given by the simple change-of-basis

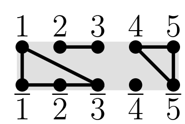
$$\mathcal{L}_\pi = \sum_{\nu \leq \pi} \mathcal{T}_\nu.$$

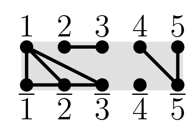
To contrast the example at the end of the previous section, the product in this basis looks like the following.

$$\mathcal{L}_{12/\bar{1}/3\bar{2}\bar{3}}\mathcal{L}_{1/\bar{1}/\bar{2}/23\bar{3}} = n\mathcal{L}_{12/\bar{1}/\bar{2}/3\bar{3}}.$$

For a set partition $\pi \in \Pi_{2r}$, there is a classical graph-theoretic representation of π on two rows of vertices with the top row being labeled 1 through r and the bottom being labeled $\overline{1}$ through $\overline{r}$. Two vertices of this graph are adjacent if and only if their labels are in the same block of π. Writing each connected component as a complete graph is visually messy—luckily, the product formula only depends on the connected components of the graph. Empowered by this fact, we define the *diagram* of a set partition π to be the equivalence class of graphs with connected components given by π.

Example 1.19. The following two graphs are representatives for the diagram of the set partition $\pi = \{\{1, \overline{1}, \overline{2}, \overline{3}\}, \{2, 3\}, \{\overline{4}\}, \{4, 5, \overline{5}\}\}$:

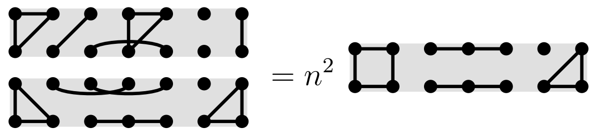




Going forward, we will not distinguish between a set partition π and its diagram. Because of this graphical representation, for a set S with elements from $[r] \cup [\overline{r}]$ we will sometimes refer to elements at the "top" of S to mean the unbarred elements and elements at the "bottom" of S to mean the barred elements.

The product formula for $\mathcal{L}_\pi$ can be stated in terms of diagrams as follows. To compute the product of $\mathcal{L}_\pi$ and $\mathcal{L}_\nu$, place a graph representing π above one representing ν and identify the vertices on the bottom of π with the corresponding vertices of ν to create a three-tiered diagram. Let $\pi \circ \nu$ be the restriction of this diagram to the very top and very bottom, preserving which vertices are connected and let $\ell(\pi, \nu)$ be the number of components entirely in the middle of the three-tier diagram. Then $\mathcal{L}_\pi \mathcal{L}_\nu = n^{\ell(\pi,\nu)} \mathcal{L}_{\pi\circ\nu}$.

Example 1.20.



There are two concepts that become immediately apparent from the diagram basis. First, we say a block of π is a *propagating block* if it has both unbarred and barred elements. Write $\mathrm{rank}(\pi)$, called the *rank* or propagating number, for the number of propagating blocks of π. The *transpose* π^T of π is the set partition obtained by reflecting the diagram of π upside-down. These concepts interact with the product structure in the following ways:

$$\pi^T \circ \nu^T = (\nu \circ \pi)^T$$
$$\mathrm{rank}(\pi \circ \nu) \leq \min\{\mathrm{rank}(\pi), \mathrm{rank}(\nu)\}$$

Subalgebras. The partition algebra has a number of interesting subalgebras obtained by restricting to diagrams with certain constraints. We summarize some of these subalgebras here because they will be useful in illustrating the extent (and limitations) of the main theorem of Chapter 2. Each of the algebras in the following table arise as a centralizer of the action of a different group G. Although we merely use these subalgebras as examples, they have a beautiful theory of their own, and we provide some references to learn more about them.

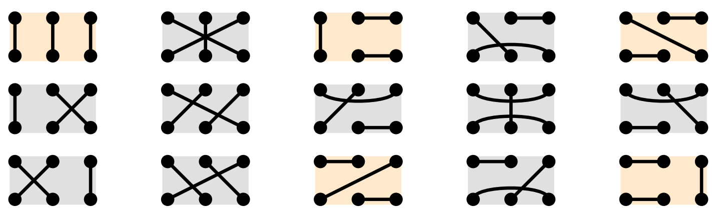
G	Centralizer	Description of blocks
GL_n	Symmetric group algebra $\mathbb{C}\mathfrak{S}_r$	Popagating of size two
O_n	Brauer Algebra $B_r(n)$ [8, 53, 22]	Size two
$\mathfrak{S}_n$	Partition Algebra $P_r(n)$	No restrictions
$U_q(\mathfrak{gl}_2)$	Temperley-Lieb $TL_r(n)$ [34, 50, 26, 54]	Noncrossing of size two

Now, we illustrate a typical element from each of these algebras:

$\mathfrak{S}_4$ $B_4(n)$ $P_4(n)$ $TL_4(n)$

The first three algebras in this table come from restricting the diagonal action of GL_n, and so they all contain the symmetric group algebra. The last one comes from an action of the quantum group $U_q(\mathfrak{gl}_2)$ and because its diagrams are noncrossing, it does not contain the symmetric group.

Example 1.21. We show the 15 elements of $B_3(n)$ when $n \geq 6$ and highlight the 5 elements of $TL_3(n)$ in orange:

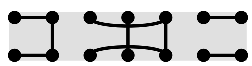
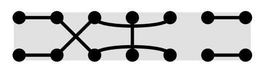


Representations. The double-centralizer theorem (Theorem 1.17) tells us that the irreducible representations $W^{\lambda}_{P_r(n)}$ of $P_r(n)$ are indexed by partitions of n, so it's reasonable to expect that its representations should have descriptions in terms of tableaux of a particular shape. In [4, 5, 40], the dimension of $W^{\lambda}_{P_r(n)}$ is computed to be the number of certain set-valued tableaux called standard set partition tableaux of shape λ. Before defining these tableaux, we give a construction of the irreducible representations of $P_r(n)$ from [20] to see how these set-valued tableaux arise naturally from a tensor product.

First, we call a diagram $w \in \Pi_{2r}$ a *symmetric m-diagram* if $w^T = w$, w has m propagating blocks, and each propagating block connects a block on the top of w

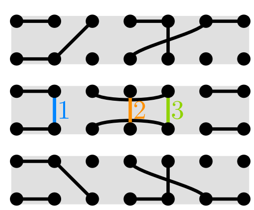
with its mirror image on the bottom (each i replaced with $\bar{i}$). Write $\mathcal{W}^m_{P_r}$ for the set of symmetric m-diagrams in Π_{2r}.

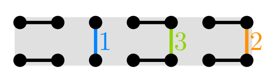
Example 1.22.

 $\in \mathcal{W}^3_{P_7}$  $\notin \mathcal{W}^3_{P_7}$

The *conjugation* of $w \in \mathcal{W}^m_{P_r}$ by $\pi \in \Pi_{2r}$ is $\pi \circ w \circ \pi^T$. Notice that the multiplicative properties of the rank and the transpose imply that $\pi \circ w \circ \pi^T \in \mathcal{W}^{m'}_{P_r}$ for some $m' \leq m$. If the conjugation $\pi \circ w \circ \pi^T$ of $w \in \mathcal{W}^m_{P_r}$ has m propagating blocks, then no propagating blocks were combined or cut off, so we can identify which block of the conjugation each block of w became (see the colors of the blocks in Example 1.23). We can record the permutation $\sigma_{\pi,w} \in \mathfrak{S}_m$ of the propagating blocks called the *twist* of the conjugation of w by π.

Example 1.23. A symmetric 3-diagram w, its conjugation by a set partition π, and the corresponding twist $\sigma_{\pi,w}$.

$\pi \circ w \circ \pi^T =$ 

$=$ 

$\sigma_{\pi,w} = 132$

With the concept of the conjugation and twist, we are now ready to construct the irreducible representations of $P_r(n)$. For a partition λ, write λ^* for the partition obtained by removing the first part of λ. Let $\Lambda^{P_r(n)} = \{\lambda \vdash n : |\lambda^*| \leq r\}$. As the notation suggests, this will be the indexing set for the irreducible $P_r(n)$-representations.

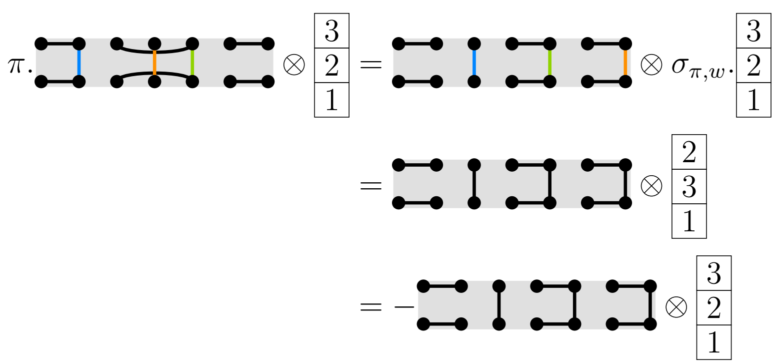
For $\lambda \in \Lambda^{P_r(n)}$, define

$$P_r^\lambda := \mathbb{C}\mathcal{W}_{P_r}^m \otimes S^{\lambda^*} = \mathbb{C}\{w \otimes n_t : w \in \mathcal{W}_{P_r}^m, t \in \mathcal{SYT}(\lambda^*)\}$$

where an element $\pi \in \Pi_{2r}$ acts on P_r^λ by

$$\pi.(w \otimes n_t) = \begin{cases} n^{\ell(\pi,w)} \pi \circ w \circ \pi^T \otimes \sigma_{\pi,w}.n_t & \operatorname{rank}(\pi \circ w \circ \pi^T) = m \\ 0 & \text{otherwise} \end{cases}.$$

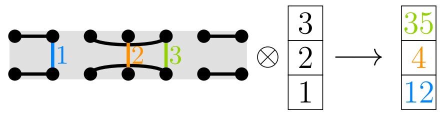
Example 1.24. The action of the set partition π in Example 1.23 on a basis element of $P_6^{(1,1,1)}$ (notice that the tableau requires straightening in the final step):



Theorem 1.25 ([20, Theorem 3.21])**.** *If $n \geq 2r$, then $\{P_r^\lambda : \lambda \in \Lambda^{P_r(n)}\}$ is a complete set of pairwise nonisomorphic irreducible $P_r(n)$ representations.*

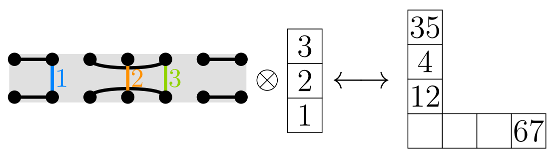
Recall that our goal is to construct $P_r(n)$ as a combinatorial action on tableaux of shape λ. Luckily, it is natural to construct a tableau from a tensor $w \otimes n_t$ by placing the top of the ith propagating block of w in last-letter order into the tableau t in the box labeled i. This creates a tableau of shape λ^* whose entries are increasing along rows and columns in the last-letter order.

Example 1.26.



We've lost some information in this process—namely the non-propagating blocks of w. To account for them, we add λ_1 boxes in as a new row at the bottom and fill the boxes at the end with the non-propagating blocks at the top of w. Note that for $\lambda \in \Lambda^{P_r(n)}$ when $n \geq 2r$, there are always enough boxes in the first row to include the non-propagating content in a way that doesn't touch the propagating content.

Example 1.27.



This construction motivates the following definition. Let ρ be a set partition of $[r]$ and $\lambda \vdash n$ such that $|\lambda^*| \leq \ell(\rho)$. A *set-partition tableau* of shape λ and content ρ is a filling T of the Young diagram with the blocks of ρ along with at least as many empty boxes in the last row as total boxes in the second row. Write $\mathcal{SPT}(\lambda, r)$ for the set of set partition tableaux of shape λ and content a set partition of $[r]$. A *standard set-partition tableau* is a set-partition tableau with rows and columns increasing as above, this time with the last-letter order. Write $\mathcal{SSPT}(\lambda, r)$ for the subset of standard tableaux in $\mathcal{SPT}(\lambda, r)$.

Example 1.28. An example of a standard set-partition tableau when $\lambda = (5, 2, 1)$ and

$\rho = \{\{2,4\},\{3,5\},\{1,7\},\{6,8\},\{9\}\}$ is given by

$$T = \begin{array}{ccccc} \cline{1-1} \multicolumn{1}{|c|}{17} & & & & \\ \cline{1-2} \multicolumn{1}{|c|}{35} & \multicolumn{1}{c|}{68} & & & \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{24} & \multicolumn{1}{c|}{9} \\ \hline \end{array} \quad .$$

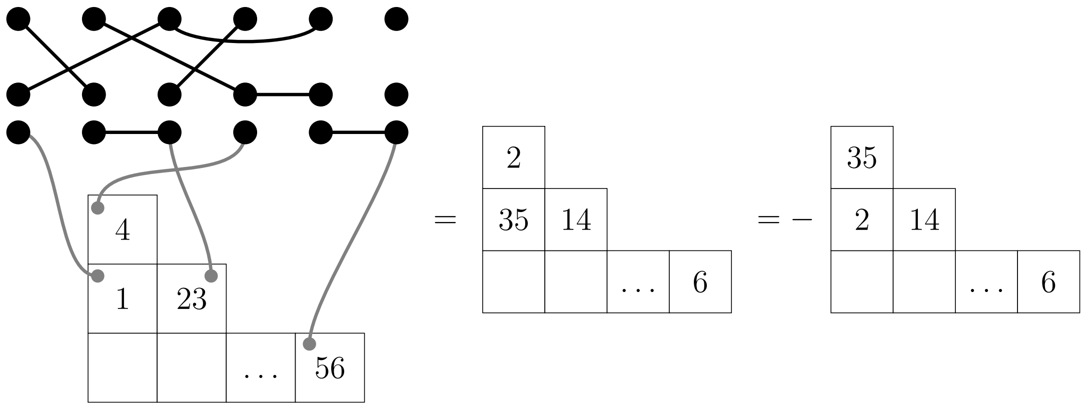
The construction above is a bijection between standard set-partition tableaux $\mathcal{SSPT}(\lambda, r)$ and tensors $w \otimes n_t \in \mathbb{C}\mathcal{W}^m_{P_r} \otimes S^{\lambda^*}$. For $T \in \mathcal{SSPT}(\lambda, r)$, write $v_T = w \otimes n_t$ for its corresponding tensor. Then

$$\{v_T : T \in \mathcal{SSPT}(\lambda, r)\}$$

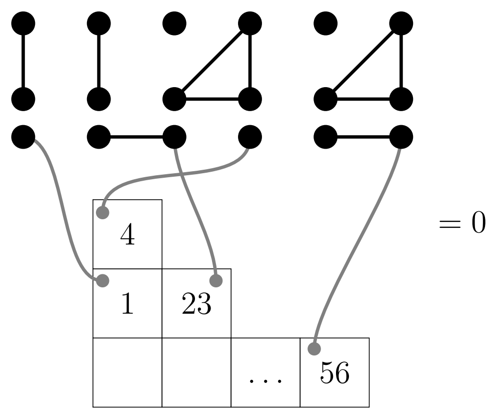
forms a basis for P_r^λ.

Finally, we want to combinatorially describe the action of a diagram $\pi \in \Pi_{2r}$ on v_T. First, we pull out the content of T as a single row and place a graph representing π on top of it. We identify the corresponding vertices and attempt to replace the content of each box above the first row with the content at the top of π it is connected to. If we fail to do this, it means that either two propagating blocks were combined or one was severed in the conjugation $\pi \circ w \circ \pi^T$, and so the result is zero. Otherwise, we empty the boxes in the first row of T and put any non-propagating blocks at the top of π instead.

Example 1.29. The following action of a set partition on a tableau results in a nonstandard tableau that must be straightened.



The action of the following set partition results in zero because the blocks $\boxed{4}$ and $\boxed{23}$ are combined.



1.3.2. The Multiset Partition Algebra

The diagram algebras discussed above arise from representations on tensor spaces. In this section, we direct our attention back to a setting with more symmetry: representations on polynomial spaces. The multiset partition algebra $\mathbb{MP}_{r,k}(x)$ was introduced in [41] as a Howe duality analog of the partition algebra. In particular, when x is specialized to an integer $n \geq 2r$, the algebra $\mathbb{MP}_{r,k}(n)$ is isomorphic to $\mathrm{End}_{\mathfrak{S}_n}(\mathcal{P}^r(V_{n,k}))$ where the action of $\mathfrak{S}_n$ is the restriction of the GL_n action of Howe duality to the $n \times n$ permutation matrices.

The Orbit Basis. First, we follow the exposition in [41] to see how combinatorial objects analogous to partition diagrams appear for the multiset partition algebra. In order to understand the centralizer $\mathrm{End}_{\mathfrak{S}_n}(\mathcal{P}^r(V_{n,k}))$, we'll first need to write down a basis for $\mathcal{P}^r(V_{n,k})$. For the tensor power $V_n^{\otimes r}$ we were able to write $e_{\boldsymbol{i}} = e_{\boldsymbol{i}_1} \otimes e_{\boldsymbol{i}_r}$, encoding the subscripts in a single sequence $\boldsymbol{i} \in [n]^r$. Because the indeterminates in $\mathcal{P}^r(V_{n,k})$ have two subscripts, we can try to mimic this notation with a pair of sequences: For sequences $\boldsymbol{i} \in [n]^r$ and $\boldsymbol{j} \in [k]^r$, write $x_{(\boldsymbol{i},\boldsymbol{j})} = x_{\boldsymbol{i}_1,\boldsymbol{j}_1} \ldots x_{\boldsymbol{i}_r,\boldsymbol{j}_r}$. In the following example, we can already see how the additional symmetry that $\mathcal{P}^r(V_{n,k})$ has complicates the situation.

Example 1.30. Two different pairs of sequences can give rise to the same monomial:

$$(1,1,2),(1,2,2) \to x_{11}x_{12}x_{22}$$
$$(1,2,1),(1,2,2) \to x_{11}x_{22}x_{12} = x_{11}x_{12}x_{22}$$

To avoid these ambiguities, we need to impose an order on our sequences. If we require that $\boldsymbol{j}_1 \leq \boldsymbol{j}_2 \leq \ldots \boldsymbol{j}_r$ and $\boldsymbol{i}_s \leq \boldsymbol{i}_t$ whenever $\boldsymbol{j}_s = \boldsymbol{j}_t$, then

$$\{x_{(\boldsymbol{i},\boldsymbol{j})} : 1 \leq \boldsymbol{i}_s \leq n \text{ and } 1 \leq \boldsymbol{j}_t \leq k\}$$

is a basis for $\mathcal{P}^r(V_{n,k})$. We can then describe the $\mathfrak{S}_n$-action on these monomials as

$$\sigma.x_{\boldsymbol{i}_1\boldsymbol{j}_1}\cdots x_{\boldsymbol{i}_r\boldsymbol{j}_r} = x_{\sigma^{-1}(\boldsymbol{i}_1)\boldsymbol{j}_1}\cdots x_{\sigma^{-1}(\boldsymbol{i}_r)\boldsymbol{j}_r}.$$

We can think of this action on the pairs of sequences as $\sigma.(\boldsymbol{i},\boldsymbol{j}) = (\sigma^{-1}(\boldsymbol{i}),\boldsymbol{j})$ where $\sigma^{-1}(\boldsymbol{i})$ is the result of applying σ^{-1} to each element of $\boldsymbol{i}$ and then possibly reordering the sequence subject to the above condition. As we did for the partition algebra, we consider the induced action on $\mathrm{End}(\mathcal{P}^r(V_{n,k}))$ by considering its effect on matrix units $E^{(\boldsymbol{i},\boldsymbol{j})}_{(\boldsymbol{i}',\boldsymbol{j}')}$ and we similarly see that we have a basis given by orbits of pairs of pairs of sequences $((\boldsymbol{i},\boldsymbol{j}),(\boldsymbol{i}',\boldsymbol{j}'))$ under the $\mathfrak{S}_n$-action.

Example 1.31. The orbit of $\boldsymbol{i} = (1,1,2)$, $\boldsymbol{i}' = (1,1,1)$ and $\boldsymbol{j} = (1,2,2)$, $\boldsymbol{j}' = (2,2,2)$ under the $\mathfrak{S}_3$-action:

$$\begin{array}{ccc}
\epsilon & (112,122) & (111,222) \\
(1\,2) & (221,122) \to (212,122) & (222,222) \\
(1\,3) & (332,122) \to (323,122) & (333,222) \\
(2\,3) & (113,122) & (111,222) \\
(1\,2\,3) & (223,122) & (222,222) \\
(1\,3\,2) & (331,122) \to (323,122) & (333,222)
\end{array}$$

Looking at the first sequence in each pair, we see set partitions arise similarly to the orbit basis for the partition algebra by grouping up the indices where values are the same into blocks. The difference now is that the second sequence in each pair tells us how the values in the first sequence can be exchanged or thought of as indistinguishable. To capture the phenomenon of indistinguishable elements we need to introduce a few definitions.

A *multiset* of size r from a set S is a collection of r unordered elements of S which can be repeated. We will write multisets in $\{\!\{,\}\!\}$ to differentiate them from sets and we will usually denote them by a capital letter with a tilde. We may write multisets using exponential notation $\tilde{S} = \{\!\{{s_1}^{m_1}, \ldots, {s_k}^{m_k}\}\!\}$ where the multiplicity of the element s_i is given by the exponent m_i. We write $m_{s_i}(\tilde{S}) = m_i$ for this multiplicity. Given multisets $\tilde{S} = \{\!\{{s_1}^{m_1}, \ldots, {s_k}^{m_k}\}\!\}$ and $\tilde{R} = \{\!\{{s_1}^{n_1}, \ldots, {s_k}^{n_k}\}\!\}$, write $\tilde{S} \uplus \tilde{R} = \{\!\{{s_1}^{m_1+n_1}, \ldots, {s_k}^{m_k+n_k}\}\!\}$ for their union.

A *multiset partition* $\tilde{\rho}$ of a multiset $\tilde{S}$ is a multiset of multisets called blocks whose union is $\tilde{S}$. We write $\ell(\tilde{\rho})$ for the number of blocks. Returning to the orbit in Example 1.31, consider the multiset $\overline{j} \uplus j' = \{\!\{\overline{1}, \overline{2}, \overline{2}, 2, 2, 2\}\!\}$. The orbit determines a multiset partition of this multiset where two elements are in the same block if and only if the corresponding values in the first sequence of the pair are equal. Here, that multiset partition is $\tilde{\pi} = \{\!\{\{\!\{2, 2, 2, \overline{1}, \overline{2}\}\!\}, \{\!\{\overline{2}\}\!\}\}\!\}$. Write $\tilde{\Pi}_{2r,k}$ for the set of multiset partitions with r elements from $[k]$ and r elements from $[\overline{k}]$. The *orbit basis* $\{\mathcal{X}_{\tilde{\pi}} : \tilde{\pi} \in \tilde{\Pi}_{2r,k}\}$ is given by

$$\mathcal{X}_{\tilde{\pi}} = \sum_{((\boldsymbol{i},\boldsymbol{j}),(\boldsymbol{i}',\boldsymbol{j}'))} E^{(\boldsymbol{i},\boldsymbol{j})}_{(\boldsymbol{i}',\boldsymbol{j}')}$$

where the sum is over the orbit corresponding to the multiset partition $\tilde{\pi}$.

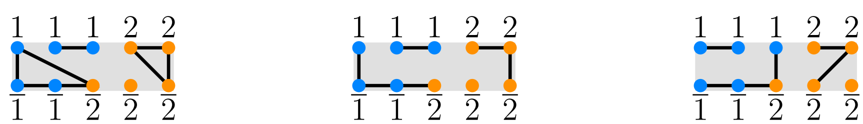
We again delay stating the product formula for this orbit basis until Appendix A, but we give the following example to illustrate the similarity of the product to that of the orbit basis for the partition algebra.

Example 1.32. The product of two orbit basis elements:

$$\mathcal{X}_{11/\overline{12}/12\overline{12}}\mathcal{X}_{12/\overline{11}/12\overline{11}} = (n-3)\mathcal{X}_{11/\overline{11}/12\overline{11}} + (n-2)\mathcal{X}_{11\overline{11}/12\overline{11}} + \mathcal{X}_{11\overline{11}/12/\overline{11}} + 2\mathcal{X}_{11/12/\overline{11}/\overline{11}}$$

Multiset Partition Diagrams. Like for set partitions, we can consider a graph-theoretic representation of any multiset partition $\tilde{\pi} \in \tilde{\Pi}_{2r,k}$. This time we place r vertices on the top labeled by the unbarred elements of the blocks of $\tilde{\pi}$ in weakly increasing order and place r vertices on the bottom labeled by the barred elements in weakly increasing order. We then connect the vertices in any way so that the labeled connected components taken together are $\tilde{\pi}$.

Example 1.33. The multiset partition $\tilde{\pi} = \{\!\!\{ \{\!\!\{1, \overline{1}, \overline{1}, \overline{2}\}\!\!\}, \{\!\!\{1, 1\}\!\!\}, \{\!\!\{\overline{2}\}\!\!\}, \{\!\!\{2, 2, \overline{2}\}\!\!\} \}\!\!\}$ could be represented by any of the following graphs:



Again we may have many graphs which represent the same multiset partition. The *diagram* of $\tilde{\pi}$ is the equivalence class of graphs whose labeled connected components give $\tilde{\pi}$.

We will often drop the labels on the diagrams. A set partition diagram will be distinguished by the black color of its vertices, and it will be understood that the vertices are labeled in increasing order left-to-right. A multiset partition diagram will be distinguished by its colored vertices. Its vertices are understood to be labeled with **blue**, **orange**, **green**, and **purple** representing 1, 2, 3, and 4 respectively.

Dimensions of Representations. Theorem 1.17 can be used to glean some information about irreducible representations of $\mathbb{MP}_{r,k}(n)$. The theorem tells us that $\mathcal{P}^r(V_{n,k})$ decomposes as a $\mathbb{C}\mathfrak{S}_n \times \mathbb{MP}_{r,k}(n)$-module as

$$\mathcal{P}^r(V_{n,k}) \cong \bigoplus_{\lambda \in \Lambda} S^\lambda \otimes W^\lambda_{\mathbb{MP}_{r,k}(n)}$$

where $\Lambda \subseteq \Lambda^{\mathfrak{S}_n}$ is a subset of partitions of n. An important consequence is that the dimension of $W^\lambda_{\mathbb{MP}_{r,k}(n)}$ is equal to the multiplicity of the representation S^λ in $\mathcal{P}^r(V_{n,k})$ as an $\mathfrak{S}_n$-module. In [41] and [38], the authors use machinery from character theory and symmetric functions to compute the multiplicities of S^λ in $\mathcal{P}^r(V_{n,k})$ as enumerating a class of multiset-valued tableaux. Let $\tilde{\rho}$ be a multiset partition from $[k]$ and $\lambda \vdash n$ such that $|\lambda^*| \leq \ell(\tilde{\rho})$. A *multiset partition tableau* of shape λ and content $\tilde{\rho}$ is a filling $\tilde{T}$ of the Young diagram of λ with the blocks of $\tilde{\rho}$ with at least as many empty boxes in the last row as total boxes in the second row. Write $\mathcal{MPT}(\lambda, r, k)$ for the set of multiset partition tableaux of shape λ with content a multiset partition from k with a total of r elements. A *semistandard multiset partition tableau* is a multiset partition tableau which weakly increases across rows and strictly increases up columns with respect to the last-letter order. Write $\mathcal{SSMPT}(\lambda, r, k)$ for the subset of semistandard tableaux in $\mathcal{MPT}(\lambda, r, k)$.

Example 1.34.

$$\begin{array}{|c|c|c|c|} \cline{1-1} 112 \\ \cline{1-1} 12 \\ \cline{1-2} 11 & 11 \\ \hline & & 22 & 3 \\ \hline \end{array} \in \mathcal{SSMPT}((4,2,1,1),12,3)$$

The dimension of an irreducible $W^\lambda_{\mathbb{MP}_{r,k}(n)}$ appearing in the above decomposition is

$$\dim(W^{\lambda}_{\mathbb{MP}_{r,k}(n)}) = |\mathcal{SSMPT}(\lambda, r, k)|.$$

Compared to the other combinatorial objects defined in this chapter, these tableaux may seem unmotivated without seeing their origins in symmetric function theory. We will revisit them in Chapter 3 in a way that makes them appear more natural.

We will see shortly in Section 1.4 that $\Lambda = \Lambda^{\mathbb{MP}_{r,k}(n)}$. That is, each irreducible $\mathbb{MP}_{r,k}(n)$-module appears in $\mathcal{P}^r(V_{n,k})$ and so we know the dimension and number of them.

Section 1.4

RSK Algorithm

In Section 1.2.1, we introduced Equation 1.1 relating the dimension of an algebra A to the dimensions of its irreducible representations. Specializing to the case $A = \mathbb{C}\mathfrak{S}_n$, we obtain the equation

$$n! = \sum_{\lambda \vdash n} |\mathcal{SYT}(\lambda)|^2.$$

This equation suggests a bijection between permutations and pairs of standard Young tableaux (P, Q) with the same shape. The RSK algorithm—named for Robinson, Schensted, and Knuth—provides such a bijection explicitly. In this section, we outline this algorithm and a generalization that will be useful to us. See [45, Chapter 3] for a careful treatment of row-insertion and the RSK algorithm.

Row Insertion. The RSK algorithm is built up from an operation on tableaux called *row insertion*. To insert an integer m into the jth row of a tableau t, we do the following.

(a) If m is greater than or equal to each entry in the jth row (in particular, if the jth row is empty), simply place m at the end of the row.

(b) Otherwise, let n be the left-most entry strictly greater than m in the jth row. Replace n with m and insert n into the $(j+1)$st row. We say that m "bumps" n to the next row.

A good way to remember which entry gets bumped is to consider what will keep the row weakly increasing. Although handling repeated entries isn't applicable in the case of standard Young tableaux, it will be very important in the generalization that follows.

Example 1.35. We insert the number 2 into the first row of the following tableau:

$$
\begin{array}{ll}
 & \begin{array}{|c|c|c|c|c|l}
\cline{1-1} 5 \\
\cline{1-2} 2 & 6 \\
\hline 1 & 2 & 3 & 3 & 4 & \leftarrow 2 \\
\hline
\end{array} \\[2ex]
= & \begin{array}{|c|c|c|c|c|}
\cline{1-1} 5 \\
\cline{1-2} 2 & \multicolumn{1}{c|}{6} & \multicolumn{3}{l}{\leftarrow 3} \\
\hline 1 & 2 & 2 & 3 & 4 \\
\hline
\end{array} \\[2ex]
= & \begin{array}{|c|c|c|c|c|}
\cline{1-1} 5 & \multicolumn{4}{l}{\leftarrow 6} \\
\cline{1-2} 2 & \multicolumn{1}{c|}{3} \\
\hline 1 & 2 & 2 & 3 & 4 \\
\hline
\end{array} \\[2ex]
= & \begin{array}{|c|c|c|c|c|}
\cline{1-2} 5 & \multicolumn{1}{c|}{6} \\
\cline{1-2} 2 & \multicolumn{1}{c|}{3} \\
\hline 1 & 2 & 2 & 3 & 4 \\
\hline
\end{array}
\end{array}
$$

Notice that if we know in which row the algorithm terminates (in the above example, the third row when 6 is placed at the end), we can easily reverse the process by "unbumping." The less obvious fact is that if we insert a positive integer into a tableau whose rows and columns are increasing, the resulting tableau's rows and columns will also be increasing.

RSK Algorithm. We treat the RSK algorithm in some generality (after [10]) because it will be useful for the algorithm for multisets in the following section. Given two totally ordered alphabets A and B, a *generalized permutation* is an array

$$\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_\ell \\ b_1 & b_2 & \cdots & b_\ell \end{pmatrix}$$

where $a_1 \leq \cdots \leq a_\ell$ and $b_i \leq b_{i+1}$ whenever $a_i = a_{i+1}$. To create a tableau from this generalized permutation, we simply start with an empty tableau and insert the values b_1 up to b_ℓ in order. We write $P_0 = \emptyset$ and $P_i = P_{i-1} \leftarrow b_i$. The final tableau $P(\sigma) = P_\ell$ is called the *insertion tableau*. This tableau does not contain enough information to recover σ—for that we need to know where the row-insertion ended at each step. The recording tableau $Q(\sigma)$ of the same shape as $P(\sigma)$ records this information by recording in each box the value a_i if that box was added at step i.

When $A = B = [n]$ and the numbers in each row are distinct, we get our usual definition of a permutation and we end up with a pair of standard Young tableaux. This is the form that Robinson's algorithm took [44]. If repetition is allowed in the second row, we recover Schensted's algorithm [46]. Knuth's version of the algorithm allowed repetition in both rows [28].

Example 1.36. The insertion and recording tableaux for $\sigma = \begin{pmatrix} 123456 \\ 162435 \end{pmatrix}$:

$$
\begin{array}{llll}
P_1 = \emptyset \leftarrow 1 = \boxed{1} & & Q_1 = \boxed{1} \\[6pt]
P_2 = \boxed{1} \leftarrow 6 = \boxed{1}\boxed{6} & & Q_2 = \boxed{1}\boxed{2} \\[6pt]
P_3 = \boxed{1}\boxed{6} \leftarrow 2 = \begin{array}{l} \boxed{6} \\ \boxed{1}\boxed{2} \end{array} & & Q_3 = \begin{array}{l} \boxed{3} \\ \boxed{1}\boxed{2} \end{array} \\[6pt]
P_4 = \begin{array}{l} \boxed{6} \\ \boxed{1}\boxed{2} \end{array} \leftarrow 4 = \begin{array}{l} \boxed{6} \\ \boxed{1}\boxed{2}\boxed{4} \end{array} & & Q_4 = \begin{array}{l} \boxed{3} \\ \boxed{1}\boxed{2}\boxed{4} \end{array} \\[6pt]
P_5 = \begin{array}{l} \boxed{6} \\ \boxed{1}\boxed{2}\boxed{4} \end{array} \leftarrow 3 = \begin{array}{l} \boxed{6} \\ \boxed{4} \\ \boxed{1}\boxed{2}\boxed{3} \end{array} & & Q_5 = \begin{array}{l} \boxed{5} \\ \boxed{3} \\ \boxed{1}\boxed{2}\boxed{4} \end{array} \\[6pt]
P_6 = \begin{array}{l} \boxed{6} \\ \boxed{4} \\ \boxed{1}\boxed{2}\boxed{3} \end{array} \leftarrow 5 = \begin{array}{l} \boxed{6} \\ \boxed{4} \\ \boxed{1}\boxed{2}\boxed{3}\boxed{5} \end{array} & & Q_6 = \begin{array}{l} \boxed{5} \\ \boxed{3} \\ \boxed{1}\boxed{2}\boxed{4}\boxed{6} \end{array}
\end{array}
$$

This algorithm establishes the desired bijection

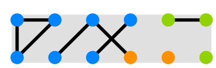
$$\mathfrak{S}_n \overset{\sim}{\longleftrightarrow} \biguplus_{\lambda \vdash n} \mathcal{SYT}(\lambda) \times \mathcal{SYT}(\lambda).$$

A Multiset Version. Now we give an analogous insertion algorithm for $\mathbb{MP}_{r,k}(n)$ introduced in [10, Section 6] which takes a multiset partition in $\tilde{\Pi}_{2r,k}$ and produces two tableaux of the same shape $\lambda \vdash n$. Fix integers $r, k > 0$ and $n \geq 2r$. Given a multiset partition $\tilde{\pi} = \{\tilde{S}_1, \ldots, \tilde{S}_\ell, \tilde{T}_1, \ldots, \tilde{T}_m\} \in \tilde{\Pi}_{2r,k}$ where the blocks $\tilde{S}_i$ are propagating and the blocks $\tilde{T}_i$ are not, write $\tilde{S}_i^+ = \tilde{S}_i \cap [k]$ and $\tilde{S}_i^- = \tilde{S}_i \cap [\overline{k}]$. Then under the last-letter order, write a generalized permutation

$$\sigma(\tilde{\pi}) = \sigma = \begin{pmatrix} \tilde{S}_1^+ & \tilde{S}_2^+ & \cdots & \tilde{S}_\ell^+ \\ \tilde{S}_1^- & \tilde{S}_2^- & \cdots & \tilde{S}_\ell^- \end{pmatrix}$$

We can then apply the RSK algorithm above to this generalized permutation to get a pair of tableaux $P(\tilde{\pi})$ and $Q(\tilde{\pi})$ of the same shape $\mu \vdash \ell$. Let $\lambda = \mu[n]$ be the result of adding a first row of length $n - \ell$ to μ to make it a partition of n. We then create a pair of tableaux of shape λ by adding a first row of empty boxes to $P(\tilde{\pi})$ and $Q(\tilde{\pi})$. Finally, we place the barred non-propagating blocks of $\tilde{\pi}$ into the empty boxes at the end of the first row added to $P(\tilde{\pi})$ and do the same for the unbarred non-propagating blocks and $Q(\tilde{\pi})$. Note that the condition that $n \geq 2r$ and the fact that $\ell \leq r$ guarantees that the resulting tableaux are multiset partition tableaux.

Example 1.37. For $r = 6, k = 3$, and $n = 12$, we compute the pair of tableaux corresponding to

$\tilde{\pi} =$  .

The propagating blocks of $\tilde{\pi}$ give the following generalized permutation:

$$\sigma(\tilde{\pi}) = \begin{pmatrix} 1 & 1 & 11 \\ 1 & 12 & 1 \end{pmatrix}.$$

Inserting this generalized permutation gives the following pair of tableaux of shape

$\mu = (2, 1)$.

$$
\begin{aligned}
P_1 &= \emptyset \leftarrow 1 = \begin{array}{|c|}\hline 1 \\ \hline\end{array} & \qquad Q_1 &= \begin{array}{|c|}\hline 1 \\ \hline\end{array} \\
P_2 &= \begin{array}{|c|}\hline 1 \\ \hline\end{array} \leftarrow 12 = \begin{array}{|c|c|}\hline 1 & 12 \\ \hline\end{array} & \qquad Q_2 &= \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \\
P_3 &= \begin{array}{|c|c|}\hline 1 & 12 \\ \hline\end{array} \leftarrow 1 = \begin{array}{|c|c}\cline{1-1} 12 & \multicolumn{1}{|c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array} & \qquad Q_3 &= \begin{array}{|c|c}\cline{1-1} 11 & \multicolumn{1}{|c}{} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array}
\end{aligned}
$$

We add the first rows including the non-propagating content to obtain tableaux of shape $\mu[12] = (9, 2, 1)$:

$$
P = \begin{array}{|c|c|c|c|c|c|c|c|c|}\cline{1-1} 12 & \multicolumn{8}{|c}{} \\ \cline{1-2} 1 & 1 & \multicolumn{7}{|c}{} \\ \hline & & & & & & & 2 & 3 \\ \hline\end{array}
\qquad
Q = \begin{array}{|c|c|c|c|c|c|c|c|c|}\cline{1-1} 11 & \multicolumn{8}{|c}{} \\ \cline{1-2} 1 & 1 & \multicolumn{7}{|c}{} \\ \hline & & & & & & & & 33 \\ \hline\end{array}
$$

This algorithm establishes a bijection

$$\tilde{\Pi}_{2r,k} \xleftrightarrow{\sim} \biguplus_{\substack{\lambda \vdash n \\ |\lambda^*| \leq r}} \mathcal{SSMPT}(\lambda, r, k) \times \mathcal{SSMPT}(\lambda, r, k).$$

This bijection tells us that the sum of squares of the dimensions of the simple modules $W^{\lambda}_{\mathbb{MP}_{r,k}(n)}$ defined in Section 1.3.2 give the total dimension of $\mathbb{MP}_{r,k}(n)$, and hence these form a complete set of simple $\mathbb{MP}_{r,k}(n)$-modules.

Chapter 2

Painted Algebras and a Diagram-Like Basis

In this section, we first introduce the painted algebra construction for an algebra B with distinguished idempotents. Then, we discuss how orbits of objects under the action of Young subgroups of the symmetric group can bridge between objects built on sets and objects built on multisets. Finally, we bring these two concepts together to analyze a family of subalgebras of $\mathbb{MP}_{r,k}(n)$ and construct a basis analogous to the diagram basis for $P_r(n)$.

Section 2.1

Painted Algebras

In order to introduce the concept of a painted algebra, we begin with a natural setting where they occur: the algebra of endomorphisms of a module which decomposes into projections by idempotents. We begin by illustrating the concept in the setting of vector spaces to establish some notation.

First, we see how the algebra of endomorphisms of a direct sum decomposes. Let U and V be n- and m-dimensional complex vector spaces respectively. Then for fixed bases of U and V, the maps in $\mathrm{Hom}_{\mathbb{C}}(U, V)$ can be thought of as $m \times n$ matrices over $\mathbb{C}$. One can obtain a basis of the direct sum $W = U \oplus V$ by simply concatenating the bases of U and V. With respect to this basis, we can realize an element of $\mathrm{End}_{\mathbb{C}}(W)$ as an $(m+n) \times (m+n)$ matrix with four blocks, each of which represents a transformation with domain and codomain each either U or V. The blocks are arranged as indicated in the following matrix:

$$\left[\begin{array}{c|c} U \to U & V \to U \\ \hline U \to V & V \to V \end{array}\right]$$

This illustrates the decomposition

$$\mathrm{End}_{\mathbb{C}}(W) \cong \mathrm{Hom}_{\mathbb{C}}(U, U) \oplus \mathrm{Hom}_{\mathbb{C}}(U, V) \oplus \mathrm{Hom}_{\mathbb{C}}(V, U) \oplus \mathrm{Hom}_{\mathbb{C}}(V, V).$$

Let $f : U \to U$, $g : U \to V$ be linear maps. To disambiguate between thinking of f and g as elements of $\mathrm{Hom}_{\mathbb{C}}(U, U)$ and $\mathrm{Hom}_{\mathbb{C}}(U, V)$ respectively versus thinking of them as elements of $\mathrm{End}_{\mathbb{C}}(W)$, we use the following notational convention. When we write gf juxtaposed, we mean their composition as maps $U \to U$ and $U \to V$. For example, $gf \in \mathrm{Hom}_{\mathbb{C}}(U, V)$ is given by $gf(u) = g(f(u))$ for $u \in U$. When we write $g \cdot f$, we mean their composition as maps in $\mathrm{End}_{\mathbb{C}}(W)$. Here, we can think about the products $f \cdot g$ and $g \cdot f$ in terms of the block matrix above. Notice that $f \cdot g = 0$ in

$\mathrm{End}_{\mathbb{C}}(W)$ because the codomain of g is not the domain of f. For example,

$$g \cdot f = \left[\begin{array}{c|c} 0 & 0 \\ \hline g & 0 \end{array}\right] \left[\begin{array}{c|c} f & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline gf & 0 \end{array}\right]$$

and

$$f \cdot g = \left[\begin{array}{c|c} f & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline g & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right].$$

Now we introduce idempotents into the story. Let $V = \mathbb{C}^3$, and consider the idempotents $e_1, e_2 \in \mathrm{End}_{\mathbb{C}}(V)$ given as follows.

$$e_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} \qquad\qquad e_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix}$$

Let $W = e_1V \oplus e_2V$. To understand how $\mathrm{End}_{\mathbb{C}}(W)$ decomposes, our analysis of direct sums above tells us that we just need to understand the spaces $\mathrm{Hom}_{\mathbb{C}}(e_iV, e_jV)$ for $i, j \in \{1, 2\}$. There is a correspondence between elements $e_j\varphi e_i \in e_j\,\mathrm{End}_{\mathbb{C}}(V)e_i$ and elements of $\mathrm{Hom}_{\mathbb{C}}(e_iV, e_jV)$ illustrated as follows:

$$e_1 \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{bmatrix} e_2 = \begin{bmatrix} 0 & 0 & 0 \\ \varphi_{21} & 0 & \varphi_{23} \\ \varphi_{31} & 0 & \varphi_{33} \end{bmatrix} \leftrightarrow \begin{bmatrix} \varphi_{21} & \varphi_{23} \\ \varphi_{31} & \varphi_{33} \end{bmatrix} \in \mathrm{Hom}_{\mathbb{C}}(e_2V, e_1V)$$

This correspondence gives an isomorphism

$$\operatorname{End}_{\mathbb{C}}(W) \cong \bigoplus_{i,j=1}^{2} e_j \operatorname{End}_{\mathbb{C}}(V) e_i.$$

If $I \in \operatorname{End}_{\mathbb{C}}(V)$ is the identity map, the matrices for $e_1 I e_1 \in \operatorname{Hom}_{\mathbb{C}}(e_1 V, e_1 V) \subset \operatorname{End}_{\mathbb{C}}(W)$ and $e_2 I e_2 \in \operatorname{Hom}_{\mathbb{C}}(e_2 V, e_2 V) \subset \operatorname{End}_{\mathbb{C}}(W)$ are as follows:

$$e_1 I e_1 \leftrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \in \operatorname{End}_{\mathbb{C}}(W) \qquad e_2 I e_2 \leftrightarrow \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \in \operatorname{End}_{\mathbb{C}}(W)$$

Notice that as elements of $\operatorname{End}_{\mathbb{C}}(W)$, we have that $(e_1 I e_1) \cdot (e_2 I e_2) = 0$ despite the fact that $e_1 e_2 \neq 0$ in $\operatorname{End}_{\mathbb{C}}(V)$.

In the following lemma, we handle the above phenomenon in more generality for V a semisimple A-module.

Lemma 2.1. *Let A be an algebra and V a semisimple A-module. Let $e_1, \ldots, e_m \in \operatorname{End}_A(V)$ be idempotents. Then*

$$\operatorname{End}_A\left(\bigoplus_{i=1}^{m} e_i V\right) \cong \bigoplus_{i,j=1}^{m} e_i \operatorname{End}_A(V) e_j$$

where the product on the right hand side of $e_i \varphi e_j \in e_i \operatorname{End}_A(V) e_j$ and $e_k \eta e_\ell \in$

$e_k \operatorname{End}_A(V)e_\ell$ *is given by*

$$(e_i\varphi e_j) \cdot (e_k\eta e_\ell) = \delta_{jk} e_i\varphi e_j\eta e_\ell \in e_i \operatorname{End}_A(V)e_\ell.$$

Proof. First, note that

$$\operatorname{End}_A\left(\bigoplus_{i=1}^m e_iV\right) \cong \bigoplus_{i,j=1}^m \operatorname{Hom}_A(e_jV, e_iV).$$

An element $e_i\varphi e_j \in e_i \operatorname{End}_A(V)e_j$ can be viewed as a map $e_jV \to e_iV$, giving rise to an injective linear map $\Phi : e_i \operatorname{End}_A(V)e_j \to \operatorname{Hom}_A(e_jV, e_iV)$. Because V is semisimple, the submodule e_jV has a complementary submodule U so that $V = e_jV \oplus U$. A map $\psi : e_jV \to e_iV \subseteq V$ can be extended to a map $\overline{\psi} : V \to V$ by setting $\overline{\psi}(u) = 0$ for all $u \in U$. Then for any $e_jv \in e_jV$, we have that $e_i\overline{\psi}e_j(e_jv) = e_i\overline{\psi}(e_jv) = e_i\psi(e_jv)$. Hence, the map Φ is also surjective, so it is an isomorphism, hence

$$\operatorname{End}_A\left(\bigoplus_{i=1}^m e_iV\right) \cong \bigoplus_{i,j=1}^m e_i \operatorname{End}_A(V)e_j.$$

For $\eta \in \operatorname{End}_A(V)$, the image of the composition $e_k\eta e_\ell$ lies in e_kV. If $j \neq k$, then this image has trivial intersection with e_jV, so $(e_i\varphi e_j) \cdot (e_k\eta e_\ell) = 0$. If instead $j = k$, we have that $(e_i\varphi e_j) \cdot (e_k\eta e_\ell) = e_i\varphi e_j e_k\eta e_\ell$. Hence,

$$\begin{aligned}(e_i\varphi e_j) \cdot (e_k\eta e_\ell) &= \delta_{jk} e_i\varphi(e_j)^2\eta e_\ell \\ &= \delta_{jk} e_i\varphi e_j\eta e_\ell \in e_i \operatorname{End}_A(V)e_\ell.\end{aligned}$$

□

This lemma motivates the following definition where B stands in for the algebra $\mathrm{End}_A(V)$ above.

Definition 2.2. For a semisimple algebra B with distinguished idempotents e_i for $i = 1, \dots, m$, the corresponding *painted algebra* with respect to these idempotents is

$$\tilde{B} = \bigoplus_{i,j=1}^{m} e_i B e_j$$

with multiplication as in the previous lemma. For a B-module M, the corresponding *painted module* with respect to these idempotents is

$$\tilde{M} = \bigoplus_{i=1}^{m} e_i M$$

for the $\tilde{B}$-module where $e_i b e_j . e_k m = \delta_{jk} e_i b e_j . m$ where the latter action is that of B on M.

To conclude this section, we show that the irreducible $\tilde{B}$-modules are precisely the painted irreducible B-modules and show how to compute their characters.

Lemma 2.3. *Let B be a semisimple algebra with distinguished idempotents e_i for $i = 1, \dots, m$. Then*

(a) For any simple B-module S, either $\tilde{S} = \{0\}$ or $\tilde{S}$ is a simple $\tilde{B}$-module.

(b) For any simple $\tilde{B}$-module T, there is a simple B-module S so that $\tilde{S} \cong T$.

Proof. Suppose $\tilde{S} \neq \{0\}$, let $\tilde{s} = \sum_{i=1}^{m} e_i s_i \in \tilde{S}$ be nonzero, and fix any $j \in [m]$ such that $e_j s_j$ is nonzero. We show that any such $\tilde{s}$ generates $\tilde{S}$ as a $\tilde{B}$-module, and so

$\tilde{S}$ is simple. Note that $e_j\tilde{s} = e_j s_j \in e_j S \subseteq S$. Because S is simple, for any $e_k s \in S$ there exists $b \in B$ such that $be_j\tilde{s} = s$ and so $e_k b e_j \tilde{s} = e_k s$. Because the elements $e_k s$ span $\tilde{S}$, we see that $\tilde{S}$ is generated by any nonzero element and hence is simple.

The remainder of the proof generalizes an argument for the case of a single idempotent found in the proof of [29, Theorem 1.10.14]. Suppose T is a simple $\tilde{B}$-module and define a B-module

$$U = \left(\bigoplus_{i=1}^{m} Ae_i\right) \otimes_{\tilde{B}} T$$

where B acts on the direct sum by left-multiplication. Because U is a module over a semisimple ring, it is Noetherian and hence has a maximal submodule. Write $S = U/M$ where M is a maximal submodule of U. Then S is a simple B-module. The goal is now to define a nonzero $\tilde{B}$-module map from T to the painted module $\tilde{S}$.

Consider the quotient map $\pi : U \to S$ and suppose $e_i \otimes t \neq 0$. We claim that $e_i \otimes t$ generates U as a B-module and hence $\pi(e_i \otimes t) \neq 0$ (else $e_i \otimes t \in M$, which would mean that $M = U$). To show this, we need only demonstrate that for any fixed $\ell \in [m]$ and $t' \in T$, there exists an element $b \in B$ such that $b.(e_i \otimes t) = e_\ell \otimes t'$. Because T is a simple module and $e_i.t \neq 0$, there exists an element $\tilde{b} = \sum_{j,k} e_j b_{j,k} e_k \in \tilde{B}$ such that

$$\begin{aligned}
\tilde{b}.e_i \otimes t &= \sum_j e_j b_{j,i} e_i.e_i \otimes t \\
&= (e_1 + \cdots + e_m) \otimes (\sum_j e_j b_{j,i} e_i.t) \\
&= (e_1 + \cdots + e_m) \otimes e_\ell t' \\
&= e_\ell \otimes t'.
\end{aligned}$$

Set $b = \sum_j e_j b_{j,i} e_i$. Then $b.e_i \otimes t = e_\ell \otimes t'$.

Let $t \in T$ be nonzero and note that in $\tilde{B}$ we have $e_1 + \cdots + e_m = 1$, so $(e_1 + \cdots + e_m).t = t$. Hence, some $e_i.t \neq 0$. Consider the map

$$e_i \pi e_i : e_i U \to e_i S.$$

Because $e_i.(e_i \otimes t) = e_i \otimes t \in e_i U$, we know that $e_i \pi e_i$ is nonzero. Define a $\tilde{B}$-module map $\bigoplus_{i=1}^m e_i U \to \bigoplus_{i=1}^m e_i S$ by $e_j r \mapsto e_j \pi(r)$. By the above observation, this is a nonzero module map. By Schur's Lemma, it is an isomorphism and hence

$$\begin{aligned} \tilde{S} &\cong \bigoplus_{i=1}^m e_i U \\ &\cong (\bigoplus_{i,j=1}^m e_i B e_j) \otimes T \\ &\cong T. \end{aligned}$$

$\square$

Lemma 2.4. *Let B be a semisimple algebra with distinguished idempotents $e_1, \ldots, e_m$ and let V be a B-module. The character of an element $e_i b e_j \in \tilde{B}$ of the corresponding painted algebra acting on $\tilde{V}$ is*

$$\chi_{\tilde{V}}(e_i b e_j) = \delta_{ij} \chi_V(e_i b).$$

Proof. Let $\rho : \tilde{B} \to \mathrm{End}(\tilde{V})$ be the representation of $\tilde{B}$ on $\tilde{V}$. Then,

$$\begin{aligned}
\chi_{\tilde{V}}(e_i b e_j) &= \mathrm{tr}(\rho(e_i b e_j)) \\
&= \mathrm{tr}(\rho(e_i b e_j e_j)) \\
&= \mathrm{tr}(\rho(e_i b e_j)\rho(e_j)) \\
&= \mathrm{tr}(\rho(e_j)\rho(e_i b e_j)) \\
&= \mathrm{tr}(\rho(e_j e_i b e_j)).
\end{aligned}$$

If $i \neq j$, then $e_j e_i = 0$, so $\chi_{\tilde{V}}(e_i b e_j) = 0$.

Now if $i = j$, then $e_i b e_i$ maps $e_i V$ back to itself and each other $e_k V$ to zero. Let $v_1, \ldots, v_\ell$ be a basis for $e_i V$ and add vectors orthogonal to $e_k V$ to complete it to a basis $v_1, \ldots, v_n$ of V. For $v = \sum_{m=1}^n c_m v_m$, write $v|_{v_m} = c_m$. Then,

$$\begin{aligned}
\chi_V(e_i b) &= \chi_V(e_i(e_i b)) \\
&= \chi_V(e_i b e_i) \\
&= \mathrm{tr}(e_i b e_i) \\
&= \sum_{m=1}^{n} (e_i b e_i)(v_m)|_{v_m} \\
&= \sum_{m=1}^{\ell} (e_i b e_i)(v_m)|_{v_m} \\
&= \chi_{\tilde{V}}(e_i b e_i).
\end{aligned}$$

$\square$

Section 2.2

Young Subgroup Orbits

When we apply the painted algebra construction to $\mathbb{MP}_{r,k}(n)$, the idempotents involved will come from Young subgroups, but we will need to extend our definition slightly. For $r, k > 0$ a *weak composition* $\boldsymbol{a}$ of r of length k is a sequence $\boldsymbol{a} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k)$ of k non-negative integers which sum to r. Write $W_{r,k}$ for the set of weak compositions of r of length k. The *Young subgroup* of $\mathfrak{S}_r$ corresponding to $\boldsymbol{a} \in W_{r,k}$ is $\mathfrak{S}_{\boldsymbol{a}} = \mathfrak{S}_{\{1,\ldots,\boldsymbol{a}_1\}} \times \cdots \times \mathfrak{S}_{\{\boldsymbol{a}_1+\cdots+\boldsymbol{a}_{k-1},\ldots,\boldsymbol{a}_1+\cdots+\boldsymbol{a}_k\}}$. The idempotents are defined by

$$s_{\boldsymbol{a}} = \frac{1}{|\mathfrak{S}_{\boldsymbol{a}}|} \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{a}}} \sigma \in \mathbb{C}\mathfrak{S}_r.$$

The symmetric group algebra $\mathbb{C}\mathfrak{S}_r$ sits naturally inside of $P_r(n)$ as the diagrams whose blocks pair one vertex on top with one on the bottom. For $\sigma \in \mathfrak{S}_r$, we will write $\mathcal{L}_\sigma$ for the diagram basis element corresponding to the set partition $\{\{\sigma(1), \overline{1}\}, \ldots \{\sigma(r), \overline{r}\}\}$. This embedding leads to natural actions on set partitions and set partition tableaux.

We've seen how orbits of $\mathfrak{S}_n$ on pairs of sequences in $[n]^r \times [n]^r$ give rise to set partitions. In this section, we see how a similar principle leads to a correspondence between multiset partitions and orbits of actions on set partitions and similarly for multiset-valued tableaux.

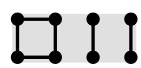
A pair of permutations $(\sigma_1, \sigma_2) \in \mathfrak{S}_r \times \mathfrak{S}_r$ can act on a set partition $\pi \in \Pi_{2r}$ by taking the product $\mathcal{L}_{\sigma_1}\mathcal{L}_\pi\mathcal{L}_{\sigma_2}$. The resulting set partition $\sigma_1.\pi.\sigma_2$ can be obtained by replacing each i in π with $\sigma_1(i)$ and each $\overline{i}$ in π with $\overline{{\sigma_2}^{-1}(i)}$.

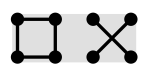
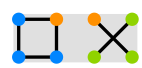
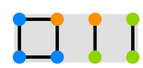
Given $\boldsymbol{a}, \boldsymbol{b} \in W_{r,k}$, define the *coloring map* $\kappa_{\boldsymbol{a},\boldsymbol{b}} : \Pi_{2r} \to \tilde{\Pi}_{2r}$ to be the function given by making the following substitutions.

$$i \mapsto \begin{cases} 1 & i \leq \boldsymbol{a}_1 \\ 2 & \boldsymbol{a}_1 < i \leq \boldsymbol{a}_1 + \boldsymbol{a}_2 \\ \vdots & \\ k & \boldsymbol{a}_1 + \cdots + \boldsymbol{a}_{k-1} < i \end{cases} \qquad \bar{i} \mapsto \begin{cases} \bar{1} & i \leq \boldsymbol{b}_1 \\ \bar{2} & \boldsymbol{b}_1 < i \leq \boldsymbol{b}_1 + \boldsymbol{b}_2 \\ \vdots & \\ \bar{k} & \boldsymbol{b}_1 + \cdots + \boldsymbol{b}_{k-1} < i \end{cases}$$

On diagrams, we can think of $\kappa_{\boldsymbol{a},\boldsymbol{b}}$ as coloring in the diagram of π with colors whose multiplicities are given by $\boldsymbol{a}$ on top and $\boldsymbol{b}$ on bottom.

Example 2.5. Two set partitions which map to the same multiset partition under the coloring map $\kappa_{(1,2,1),(2,0,2)}$:

$\kappa_{(1,2,1),(2,0,2)}\Big($  $\Big) =$ 

$\kappa_{(1,2,1),(2,0,2)}\Big($  $\Big) =$  $=$ 

Note that if π and π' are in the same $\mathfrak{S}_{\boldsymbol{a}} \times \mathfrak{S}_{\boldsymbol{b}}$-orbit, they differ by a permutation that takes each label to another label colored the same under applying $\kappa_{\boldsymbol{a},\boldsymbol{b}}$, so $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi')$. Hence, $\kappa_{\boldsymbol{a},\boldsymbol{b}}$ induces a map $\overline{\kappa}_{\boldsymbol{a},\boldsymbol{b}} : \Pi_{2r}/(\mathfrak{S}_{\boldsymbol{a}} \times \mathfrak{S}_{\boldsymbol{b}}) \to \tilde{\Pi}_{2r,k}$. Conversely, if $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi')$, then π and π' are in the same orbit. Hence, the map $\overline{\kappa}_{\boldsymbol{a},\boldsymbol{b}}$ is injective. Given $\tilde{\pi} \in \tilde{\Pi}_{2r,k}$ whose unbarred and barred multiplicities are given by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, we can easily create a set partition $\pi \in \Pi_{2r}$ such that $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \tilde{\pi}$ by simply taking any graph representing $\tilde{\pi}$ and forgetting the data of the colored vertices. Hence, the maps $\overline{\kappa}_{\boldsymbol{a},\boldsymbol{b}}$ taken together as a map $\biguplus_{\boldsymbol{a},\boldsymbol{b}\in W_{r,k}} \Pi_{2r}/(\mathfrak{S}_{\boldsymbol{a}} \times \mathfrak{S}_{\boldsymbol{b}}) \to \tilde{\Pi}_{2r,k}$ gives a bijection. This gives us a correspondence between multiset partitions and orbits of set partitions under an action of a pair of Young subgroups:

$$\tilde{\Pi}_{2r,k} \overset{\sim}{\longleftrightarrow} \biguplus_{\boldsymbol{a},\boldsymbol{b}\in W_{r,k}} \Pi_{2r}/(\mathfrak{S}_{\boldsymbol{a}} \times \mathfrak{S}_{\boldsymbol{b}}) \tag{2.1}$$

We obtain an action on tableaux from the module structure of P_r^{λ}. A permutation $\sigma \in \mathfrak{S}_r$ acts on the set $\mathcal{SPT}(\lambda, r)$ by replacing each entry i of a tableau T with $\sigma(i)$. For example,

$$(1\,3\,2)(4).\begin{array}{|c|c|} \cline{1-1} 23 & \multicolumn{1}{c}{} \\ \hline 1 & 4 \\ \hline & \\ \hline \end{array} = \begin{array}{|c|c|} \cline{1-1} 12 & \multicolumn{1}{c}{} \\ \hline 3 & 4 \\ \hline & \\ \hline \end{array}.$$

Like above, we define a surjective coloring map $\kappa_{\boldsymbol{a}} : \mathcal{SPT}(\lambda, r) \to \mathcal{MPT}(\lambda, r, k)$ for $\boldsymbol{a} \in W_{r,k}$ which replaces the numbers $\{1, \ldots, \boldsymbol{a}_1\}$ with 1, $\{\boldsymbol{a}_1 + 1, \ldots, \boldsymbol{a}_1 + \boldsymbol{a}_2\}$ with 2, etc..

Example 2.6. Two set partition tableaux that are sent to the same multiset partition tableau by the coloring map $\kappa_{(3,1)}$:

$$\kappa_{(3,1)}\left(\begin{array}{|c|c|} \cline{1-1} 23 & \multicolumn{1}{c}{} \\ \hline 1 & 4 \\ \hline & \\ \hline \end{array}\right) = \kappa_{(3,1)}\left(\begin{array}{|c|c|} \cline{1-1} 12 & \multicolumn{1}{c}{} \\ \hline 3 & 4 \\ \hline & \\ \hline \end{array}\right) = \begin{array}{|c|c|} \cline{1-1} 11 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline & \\ \hline \end{array}$$

By an analogous argument to the case of set partitions, $\kappa_{\boldsymbol{a}}(T) = \kappa_{\boldsymbol{a}}(S)$ if and only if T and S are in the same $\mathfrak{S}_{\boldsymbol{a}}$-orbit, so we get a bijection

$$\mathcal{MPT}(\lambda, r, k) \overset{\sim}{\longleftrightarrow} \biguplus_{\boldsymbol{a}\in W_{r,k}} \mathcal{SPT}(\lambda, r)/\mathfrak{S}_{\boldsymbol{a}}.$$

Remark 2.7. The orbit of a standard set partition tableau T corresponds to a multiset partition tableau $\tilde{T}$ whose rows and columns weakly increase. That is, $\tilde{T}$ is semistandard except for possible repeats within columns.

Section 2.3

Painted Diagram Algebras

In this section, we consider a subgroup G of GL_n which contains the permutation matrices $\mathfrak{S}_n \subseteq G \subseteq GL_n$. The centralizer $\mathrm{End}_G(V_n{}^{\otimes r})$ is a subalgebra $\mathfrak{S}_r \subseteq A_r(n) \subseteq P_r(n)$, so we can construct the painted algebra $\tilde{A}_{r,k}(n)$ relative to the idempotents $\{s_{\boldsymbol{a}} : \boldsymbol{a} \in W_{r,k}\}$. We will show that the centralizer $\mathrm{End}_G(\mathcal{P}^r(V_{n,k}))$ is isomorphic to $\tilde{A}_{r,k}(n)$ and use the results of Section 2.2 to describe a basis for $\tilde{A}_{r,k}(n)$ analogous to the diagram basis for the partition algebra.

Let $U_{\boldsymbol{a}}$ be the span of monomials of the form $x_{i_1 j_1} \cdots x_{i_r j_r}$ where for each $1 \leq m \leq k$, exactly $\boldsymbol{a}_m$ of the values $j_1, \ldots, j_r$ are equal to m. For example, the monomials $x_{11}x_{21}x_{23}$ and $x_{21}x_{21}x_{23}$ are both in $U_{(2,0,1)} \subset \mathcal{P}^3(V_{2,3})$. Given a matrix $M = (m_{ij}) \in GL_n$, its inverse acts on an element of $U_{\boldsymbol{a}}$ as follows:

$$\begin{aligned} M^{-1}.x_{ij} &= \sum_{\ell=1}^{n} m_{i\ell} x_{\ell j} \\ M^{-1}.x_{i_1 j_1} \ldots x_{i_r j_r} &= \left(\sum_{\ell_1=1}^{n} m_{i_1 \ell_1} x_{\ell_1 j_1}\right) \cdots \left(\sum_{\ell_r=1}^{n} m_{i_r \ell_r} x_{\ell_r j_r}\right) \\ &= \sum_{\ell_1,\ldots,\ell_r=1}^{n} m_{i_1 \ell_1} \cdots m_{i_r \ell_r} (x_{\ell_1 j_1} \ldots x_{\ell_r j_r}) \end{aligned} \tag{2.2}$$

Note that each monomial in the resulting polynomial is still in $U_{\boldsymbol{a}}$, so the subspace

$U_{\boldsymbol{a}}$ is in fact a GL_n-submodule of $\mathcal{P}^r(V_{n,k})$. This gives a decomposition

$$\mathcal{P}^r(V_{n,k}) = \bigoplus_{\boldsymbol{a}\in W_{r,k}} U_{\boldsymbol{a}}$$

as a GL_n-module. We now use this decomposition to construct a linear isomorphism $\Phi : \bigoplus_{\boldsymbol{a}\in W_{r,k}} s_{\boldsymbol{a}} {V_n}^{\otimes r} \to \mathcal{P}^r(V_{n,k})$.

For $\boldsymbol{a} \in W_{r,k}$, define a linear map $\Phi_{\boldsymbol{a}} : {V_n}^{\otimes r} \to U_{\boldsymbol{a}}$ by

$$\Phi_{\boldsymbol{a}}(e_{i_1} \otimes \cdots \otimes e_{i_r}) = \prod_{m=1}^{\boldsymbol{a}_1} x_{i_m 1} \prod_{m=\boldsymbol{a}_1+1}^{\boldsymbol{a}_1+\boldsymbol{a}_2} x_{i_m 2} \cdots \prod_{m=\boldsymbol{a}_1+\cdots+\boldsymbol{a}_{k-1}+1}^{r} x_{i_m k}.$$

For example,

$$\Phi_{(1,2,2)}(e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2) = x_{21}x_{22}x_{12}x_{13}x_{23}.$$

It's clear that $\Phi_{\boldsymbol{a}}$ is surjective. Suppose that $\Phi_{\boldsymbol{a}}(e_{\boldsymbol{i}}) = \Phi_{\boldsymbol{a}}(e_{\boldsymbol{i}'})$. This occurs exactly when $e_{\boldsymbol{i}'}$ can be obtained from $e_{\boldsymbol{i}}$ by rearranging factors grouped into the same product above (entries of $\boldsymbol{i}$ that correspond to indeterminates x_{ij} with the same second label). That is, $e_{\boldsymbol{i}'} = \sigma(e_{\boldsymbol{i}})$ for some $\sigma \in \mathfrak{S}_{\boldsymbol{a}}$. Hence, $\Phi_{\boldsymbol{a}}(e_{\boldsymbol{i}}) = \Phi_{\boldsymbol{a}}(e_{\boldsymbol{i}'})$ if and only if $s_{\boldsymbol{a}}(e_{\boldsymbol{i}}) = s_{\boldsymbol{a}}(e_{\boldsymbol{i}'})$. Then $\Phi_{\boldsymbol{a}}$ restricts to an isomorphism $s_{\boldsymbol{a}}{V_n}^{\otimes r} \xrightarrow{\sim} U_{\boldsymbol{a}}$, so the map $\Phi : \bigoplus_{\boldsymbol{a}\in W_{r,k}} s_{\boldsymbol{a}} {V_n}^{\otimes r} \to \mathcal{P}^r(V_{n,k})$ which sends $s_{\boldsymbol{a}}(e_{\boldsymbol{i}})$ to $\Phi_{\boldsymbol{a}}(s_{\boldsymbol{a}}(e_{\boldsymbol{i}}))$ is an isomorphism.

Lemma 2.8. *The linear isomorphism $\Phi : \bigoplus_{\boldsymbol{a}\in W_{r,k}} s_{\boldsymbol{a}} {V_n}^{\otimes r} \to \mathcal{P}^r(V_{n,k})$ above induces an isomorphism of algebras*

$$\mathrm{End}_G(\mathcal{P}^r(V_{n,k})) \xrightarrow{\sim} \mathrm{End}_G\left(\bigoplus_{\boldsymbol{a}} s_{\boldsymbol{a}} {V_n}^{\otimes r}\right)$$

for each subgroup G of GL_n.

Proof. The action of $M \in GL_n$ on $s_{\boldsymbol{a}}(e_{i_1} \otimes \cdots \otimes e_{i_r})$ is given by

$$\begin{aligned} M.s_{\boldsymbol{a}}(e_{i_1} \otimes \cdots \otimes e_{i_r}) &= s_{\boldsymbol{a}}(Me_{i_1} \otimes \cdots \otimes Me_{i_r}) \\ &= \sum_{\ell_1,\dots,\ell_r=1}^{n} m_{i_1\ell_r} \cdots m_{i_r\ell_r} s_{\boldsymbol{a}}(e_{\ell_1} \otimes \cdots \otimes e_{\ell_r}). \end{aligned}$$

Comparing this computation with Equation (2.2), we see that the map Φ is *nearly* a homomorphism of GL_n-modules, but the action of $M \in GL_n$ on one space is the action of M^{-1} on the other. That is, for $M \in GL_n$,

$$\Phi M = M^{-1}\Phi.$$

Multiplying by Φ^{-1} on the left and right of both sides yields an analogous statement for Φ^{-1}:

$$M\Phi^{-1} = \Phi^{-1}M^{-1}.$$

The linear isomorphism Φ induces an algebra isomorphism

$$\begin{aligned} \mathrm{End}_{\mathbb{C}}(\mathcal{P}^r(V_{n,k})) &\xrightarrow{\sim} \mathrm{End}_{\mathbb{C}}\left(\bigoplus_{\boldsymbol{a}\in W_{r,k}} s_{\boldsymbol{a}}{V_n}^{\otimes r}\right) \\ \varphi &\longmapsto \Phi^{-1}\varphi\Phi. \end{aligned}$$

Now we make the following observation for $G \subseteq GL_n$ a subgroup:

$$\begin{aligned}
\varphi \in \mathrm{End}_G(\mathcal{P}^r(V_{n,k})) &\iff \varphi = M^{-1}\varphi M && \forall M \in G \\
&\iff \Phi^{-1}\varphi\Phi = \Phi^{-1}M^{-1}\varphi M\Phi && \forall M \in G \\
&\iff \Phi^{-1}\varphi\Phi = M\Phi^{-1}\varphi\Phi M^{-1} && \forall M \in G \\
&\iff \Phi^{-1}\varphi\Phi \in \mathrm{End}_G\left(\bigoplus_{\boldsymbol{a}\in W_{r,k}} s_{\boldsymbol{a}}{V_n}^{\otimes r}\right)
\end{aligned}$$

Hence, the map $\varphi \mapsto \Phi^{-1}\varphi\Phi$ restricts to an isomorphism

$$\mathrm{End}_G(\mathcal{P}^r{V_{n,k}}^{\otimes r}) \xrightarrow{\sim} \mathrm{End}_G\left(\bigoplus_{\boldsymbol{a}\in W_{r,k}} s_{\boldsymbol{a}}{V_n}^{\otimes r}\right).$$

□

Let G be a subgroup of GL_n. Because $G \subseteq GL_n$, we have that

$$\mathbb{C}\mathfrak{S}_n \cong \mathrm{End}_{GL_n}({V_n}^{\otimes r}) \subseteq \mathrm{End}_G({V_n}^{\otimes r}).$$

Hence, the idempotents $s_{\boldsymbol{a}}$ for $\boldsymbol{a} \in W_{r,k}$ are in $\mathrm{End}_G({V_n}^{\otimes r})$, allowing us to use Lemma 2.1 to make the following computation:

$$\begin{aligned}
\mathrm{End}_G\left(\mathcal{P}^r(V_{n,k})\right) &\cong \mathrm{End}_G\left(\bigoplus_{\boldsymbol{a}\in W_{r,k}} s_{\boldsymbol{a}}{V_n}^{\otimes r}\right) \\
&\cong \bigoplus_{\boldsymbol{a},\boldsymbol{b}\in W_{r,k}} s_{\boldsymbol{a}}\,\mathrm{End}_G({V_n}^{\otimes r})s_{\boldsymbol{b}}
\end{aligned}$$

where the product is given by

$$(s_{\boldsymbol{a}}\varphi s_{\boldsymbol{b}}) \cdot (s_{\boldsymbol{c}}\psi e_{\boldsymbol{d}}) = \delta_{\boldsymbol{b},\boldsymbol{c}} s_{\boldsymbol{a}}\varphi s_{\boldsymbol{b}}\psi e_{\boldsymbol{d}}. \tag{2.3}$$

Theorem 2.9. *Let $G \subseteq GL_n$ be a subgroup and let $A_r(n) = \operatorname{End}_G\left({V_n}^{\otimes r}\right)$. Then*

$$\operatorname{End}_G\left(\mathcal{P}^r(V_{n,k})\right) \cong \tilde{A}_{r,k}(n).$$

$$\begin{tikzcd}
P_r(n) \arrow[dr, dash] & & GL_n \arrow[dr, dash] & & \mathrm{MP}_{r,k}(n) \\
A_r(n) \arrow[u, hook] \arrow[r, dash] & {V_n}^{\otimes r} \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & G \arrow[u, hook] \arrow[r, dash] & \mathcal{P}^r(V_{n,k}) \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & \tilde{A}_{r,k}(n) \arrow[u, hook] \\
S_r \arrow[u, hook] \arrow[ur, dash] & & S_n \arrow[u, hook] \arrow[ur, dash] & & GL_k \arrow[u, hook]
\end{tikzcd}$$

Figure 2.1: This diagram represents what Theorem 2.9 adds to the classical picture of Schur-Weyl duality and Howe duality. Note that the diagonal lines should only be taken to indicate mutual centralizers when the double centralizer theorem applies.

Now, we use this isomorphism to construct a basis for $\mathrm{MP}_{r,k}(n) \cong \operatorname{End}_{\mathfrak{S}_n}(\mathcal{P}^r(V_{n,k}))$ from the diagram basis of $P_r(n)$. Such a basis can be similarly constructed for other subalgebras of $P_r(n)$ which contain $\mathfrak{S}_r$. In general, the projection $s_{\boldsymbol{a}} L_\pi s_{\boldsymbol{b}}$ can be computed as follows:

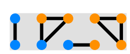
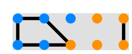
$$\begin{aligned} s_{\boldsymbol{a}}\mathcal{L}_\pi s_{\boldsymbol{b}} &= \frac{1}{|\mathfrak{S}_{\boldsymbol{a}} \times \mathfrak{S}_{\boldsymbol{b}}|} \sum_{(\sigma,\sigma') \in \mathfrak{S}_{\boldsymbol{a}} \times \mathfrak{S}_{\boldsymbol{b}}} \mathcal{L}_\sigma \mathcal{L}_\pi \mathcal{L}_{\sigma'} \\ &= \frac{1}{|\mathfrak{S}_{\boldsymbol{a}} \times \mathfrak{S}_{\boldsymbol{b}}|} \sum_{(\sigma,\sigma') \in \mathfrak{S}_{\boldsymbol{a}} \times \mathfrak{S}_{\boldsymbol{b}}} \mathcal{L}_{\sigma.\pi.\sigma'}. \end{aligned}$$

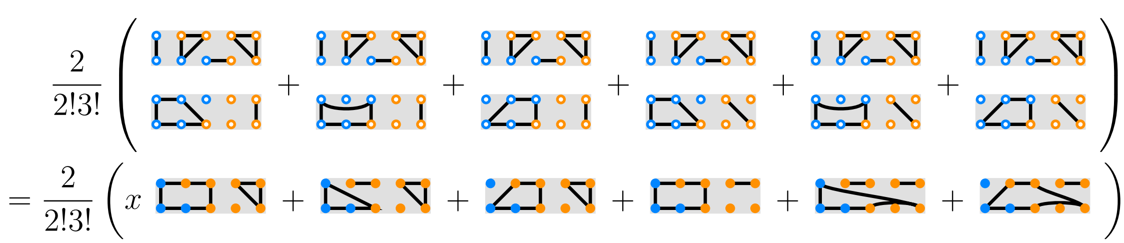
Each diagram basis element $\mathcal{L}_\pi$ projects to the sum of the orbit of π under the $\mathfrak{S}_{\boldsymbol{a}} \times \mathfrak{S}_{\boldsymbol{b}}$-action. Because the orbits are disjoint, the set of distinct projections is linearly independent and hence form a basis of $s_{\boldsymbol{a}}P_r(n)s_{\boldsymbol{b}}$. Due to the correspondence between orbits of set partitions under a Young subgroup action and multiset partitions given in Section 2.2, we know that these basis elements are indexed by multiset partitions obtained by coloring in the elements of Π_{2r} with colors whose multiplicities are given by $\boldsymbol{a}$ on the top and $\boldsymbol{b}$ on the bottom.

For $\tilde{\pi} \in \tilde{\Pi}_{2r,k}$ define $\mathcal{D}_{\tilde{\pi}} = s_{\boldsymbol{a}}\mathcal{L}_\pi s_{\boldsymbol{b}}$ where π is any set partition in the orbit corresponding to $\tilde{\pi}$. We then see that $\{\mathcal{D}_{\tilde{\pi}} : \tilde{\pi} \in \tilde{\Pi}\}$ is a basis for $\tilde{P}_{r,k}(n) \cong \mathrm{MP}_{r,k}(n)$. Using the formula in Lemma 2.1, the product $\mathcal{D}_{\tilde{\pi}}\mathcal{D}_{\tilde{\nu}}$ for $\tilde{\pi} \in \tilde{\Pi}_{2r,k}$ with multiplicities on top and bottom given by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively and $\tilde{\nu} \in \tilde{\Pi}_{2r,k}$ with multiplicities on top and bottom given by $\boldsymbol{b}'$ and $\boldsymbol{c}$ respectively is the following.

$$\begin{aligned} \mathcal{D}_{\tilde{\pi}}\mathcal{D}_{\tilde{\nu}} &= (s_{\boldsymbol{a}}\mathcal{L}_\pi s_{\boldsymbol{b}}) \cdot (s_{\boldsymbol{b}'}\mathcal{L}_\nu s_{\boldsymbol{c}}) \\ &= \delta_{\boldsymbol{b},\boldsymbol{b}'} s_{\boldsymbol{a}}\mathcal{L}_\pi s_{\boldsymbol{b}}\mathcal{L}_\nu s_{\boldsymbol{c}} \\ &= \frac{\delta_{\boldsymbol{b},\boldsymbol{b}'}}{|\mathfrak{S}_{\boldsymbol{b}}|} \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} s_{\boldsymbol{a}}\mathcal{L}_\pi \mathcal{L}_\sigma \mathcal{L}_\nu s_{\boldsymbol{c}} \\ &= \frac{\delta_{\boldsymbol{b},\boldsymbol{b}'}}{|\mathfrak{S}_{\boldsymbol{b}}|} \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} s_{\boldsymbol{a}}\mathcal{L}_\pi \mathcal{L}_{\sigma.\nu} s_{\boldsymbol{c}} \end{aligned}$$

To interpret this product combinatorially, it will be helpful to assign a combinatorial object to each term in the sum. For a pair $\tilde{\pi}, \tilde{\nu} \in \tilde{\Pi}_{2r,k}$, a *snapshot* is a pair (π, ν) where $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \tilde{\pi}$ and $\kappa_{\boldsymbol{b},\boldsymbol{c}} = \tilde{\nu}$. We can represent these visually as a stack of partition diagrams whose vertices are painted from k colors. To differentiate from the multiset partition diagrams (and to emphasize that the identically colored vertices in this situation are not interchangeable and are instead fixed in place), we draw the vertices as open circles. The formula above can then be thought of as beginning with any snapshot (π, ν) and them summing over the snapshots $\{(\pi, \sigma.\nu) : \sigma \in \mathfrak{S}_{\boldsymbol{b}}\}$. In the summand, $\mathcal{L}_\pi \mathcal{L}_{\sigma.\nu}$ is the product of the two set partitions as elements of $P_r(n)$ and multiplying by the idempotents $s_{\boldsymbol{a}}$ and $s_{\boldsymbol{b}}$ projects to the diagram-like basis element corresponding to the multiset partition obtained by filling in the vertices.

Example 2.10. The product of  and  is given by choosing a snapshot, then acting on the top of the second diagram with each permutation in $\mathfrak{S}_{(3,2)}$ Because each of these show up twice (acting first by the transposition (1 2) leaves the second diagram fixed), we put a coefficient of 2 out front to account for these duplicates.



We call this basis $\{D_{\tilde{\pi}} : \tilde{\pi} \in \tilde{\Pi}_{2r,k}\}$ for $\mathrm{MP}_{r,k}(n) \cong \mathrm{End}_{\mathfrak{S}_n}(\mathcal{P}^r(V_{n,k}))$ the *diagram-like basis*. The change-of-basis from Orellana and Zabrocki's orbit basis is given in Appendix A.

Remark 2.11. A necessary assumption in Theorem 2.9 is that the action of G is a restriction of the diagonal action of GL_n on ${V_n}^{\otimes r}$. Consequently, the centralizing algebra $A_r(n) = \mathrm{End}_G({V_n}^{\otimes r})$ must contain the symmetric group $\mathfrak{S}_r$. This means that the theorem does not apply to the planar algebras discussed in Section 1.3.1 although it seems possible that subalgebras analogous to e.g. the Temperley-Lieb algebra could appear centralizing some action of $U_q(\mathfrak{gl}_2)$ on $\mathcal{P}^r(V_{2,k})$.

Chapter 3

Irreducible Representations of $\mathbb{MP}_{r,k}(n)$

In this section, we construct the irreducible representations of $\mathbb{MP}_{r,k}(n)$. Part (ii) of Lemma 2.3 tells us that in order to construct each of these irreducible representations, we need only consider the irreducible $P_r(n)$ representations painted with respect to the idempotents $\{s_{\boldsymbol{a}} : \boldsymbol{a} \in W_{r,k}\}$. For $\lambda \in \Lambda^{P_r(n)}$, define

$$\mathbb{MP}_{r,k}^{\lambda} := \tilde{P_r^{\lambda}} = \bigoplus_{\boldsymbol{a} \in W_{r,k}} s_{\boldsymbol{a}} P_r^{\lambda}.$$

By Lemma 2.3, each module in $\{\mathbb{MP}_{r,k}^{\lambda} : \lambda \in \Lambda^{P_r(n)}\}$ is either a simple $\mathbb{MP}_{r,k}(n)$-module or the zero module, and each simple $\mathbb{MP}_{r,k}(n)$ module appears in the set. To investigate the structure of these modules, we note that for $T \in \mathcal{SPT}(\lambda, r)$, the projection

$$s_{\boldsymbol{a}} v_T = \frac{1}{|\mathfrak{S}_{\boldsymbol{a}}.T|} \sum_{S \in \mathfrak{S}_{\boldsymbol{a}}.T} v_S$$

is the average over the $\mathfrak{S}_{\boldsymbol{a}}$-orbit of T. This orbit corresponds to $\tilde{T} = \kappa_{\boldsymbol{a}}(T) \in \mathcal{MPT}(\lambda, r, k)$, and so we define

$$w_{\tilde{T}} = s_{\boldsymbol{a}} v_T = \frac{1}{\left|\kappa_{\boldsymbol{a}}^{-1}(\tilde{T})\right|} \sum_{T \in \kappa_{\boldsymbol{a}}^{-1}(\tilde{T})} v_T.$$

Lemma 3.1. *If $T \in \mathcal{SSPT}(\lambda, r)$, then either $w_{\tilde{T}} = 0$ or $\tilde{T} \in \mathcal{SSMPT}(\lambda, r, k)$.*

Proof. As observed in Section 2.2, if $T \in \mathcal{SSPT}(\lambda, r)$, then $\tilde{T}$ has rows and columns weakly increasing. If $\tilde{T}$ is not semistandard, then it must have a repeated entry within a column. Suppose $\tilde{T}$ has two boxes in the same column with the same content. For a tableau T, write T' for the tableau obtained by swapping the content of these two boxes. Then

$$w_{\tilde{T}} = w_{\tilde{T}'} = \frac{1}{|\mathfrak{S}_{\boldsymbol{a}}.T|} \sum_{S \in \mathfrak{S}_{\boldsymbol{a}}.T} v_{S'} = \frac{1}{|\mathfrak{S}_{\boldsymbol{a}}.T|} \sum_{S \in \mathfrak{S}_{\boldsymbol{a}}.T} -v_S = -w_{\tilde{T}}.$$

$\square$

We can now use Lemma 3.1 to describe a straightening algorithm for $\mathbb{MP}_{r,k}^{\lambda}$. Suppose $\tilde{T}$ is not semistandard and $w_{\tilde{T}} \neq 0$. Then there exists T in the $\mathfrak{S}_{\boldsymbol{a}}$-orbit corresponding to $\tilde{T}$ which is not standard. Then using the straightening algorithm for P_r^{λ}, we can write

$$v_T = \sum_{S \in \mathcal{SSPT}(\lambda, r)} c_S v_S.$$

Then projecting by $s_{\boldsymbol{a}}$, we obtain

$$w_{\tilde{T}} = s_{\boldsymbol{a}} v_T = \sum_{S \in \mathcal{SSPT}(\lambda,r)} c_S s_{\boldsymbol{a}} v_S = \sum_{S \in \mathcal{SSPT}(\lambda,r)} c_S w_{\tilde{S}}$$

where each $\tilde{S}$ for which $w_{\tilde{S}} \neq 0$ is semistandard.

Theorem 3.2. *The set* $\{\mathbb{MP}^{\lambda}_{r,k} : \lambda \in \Lambda^{\mathbb{MP}_{r,k}(n)}\}$ *forms a complete set of irreducible representations for* $\mathbb{MP}_{r,k}(n)$. *Furthermore, the set* $\{w_{\tilde{T}} : \tilde{T} \in \mathcal{SSMPT}(\lambda, r, k)\}$ *forms a basis for* $\mathbb{MP}^{\lambda}_{r,k}$.

Proof. Because $\mathbb{MP}^{\lambda}_{r,k}$ is the span of $\{w_{\tilde{T}} : \tilde{T} \in \mathcal{SSMPT}(\lambda, r, k)\}$, we know that $\mathbb{MP}^{\lambda}_{r,k} = 0$ unless $\lambda \in \Lambda^{\mathbb{MP}_{r,k}(n)}$. We then have that each of the $\left|\Lambda^{\mathbb{MP}_{r,k}(n)}\right|$ irreducible representations appear in the smaller set $\{\mathbb{MP}^{\lambda}_{r,k} : \lambda \in \Lambda^{\mathbb{MP}_{r,k}(n)}\}$. Because there are only $\left|\Lambda^{\mathbb{MP}_{r,k}(n)}\right|$ representations in this set, it must be a complete set of irreducible representations for $\mathbb{MP}_{r,k}(n)$.

A priori, we don't know that these $w_{\tilde{T}}$ are linearly independent, so we can only conclude that $\dim(\mathbb{MP}_{r,k}) \leq |\mathcal{SSMPT}(\lambda, r, k)|$. However, we do know that

$$\sum_{\lambda \in \Lambda^{\mathbb{MP}_{r,k}(n)}} (\dim(\mathbb{MP}^{\lambda}_{r,k}))^2 = \dim(\mathbb{MP}_{r,k}(n)) = \sum_{\lambda \in \Lambda^{\mathbb{MP}_{r,k}(n)}} |\mathcal{SSMPT}(\lambda, r, k)|^2 .$$

Hence $\dim(\mathbb{MP}_{r,k}(n)) = \#\mathcal{SSMPT}_{\lambda,r,k}$ and so the set $\{w_{\tilde{T}} : \tilde{T} \in \mathcal{SSMPT}(\lambda, r, k)\}$ indeed forms a basis of $\mathbb{MP}^{\lambda}_{r,k}$. $\square$

We now consider the action of an element $\mathcal{D}_{\tilde{\pi}}$ on $w_{\tilde{T}}$. Suppose that the multiplicities of colors in $\tilde{\pi}$ are given by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively and that the multiplicities of elements in $\tilde{T}$ are given by $\boldsymbol{c}$. Then $\mathcal{D}_{\tilde{\pi}}$ acts by

$$\begin{aligned}\mathcal{D}_{\tilde{\pi}}.w_{\tilde{T}} &= s_{\boldsymbol{a}} L_\pi s_{\boldsymbol{b}}.s_{\boldsymbol{c}} v_T \\ &= \delta_{\boldsymbol{b},\boldsymbol{c}}(s_{\boldsymbol{a}} L_\pi s_{\boldsymbol{b}} v_T) \\ &= \delta_{\boldsymbol{b},\boldsymbol{c}} \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} s_{\boldsymbol{a}} L_{\pi.\sigma} v_T.\end{aligned}$$

We can interpret this formula for diagrams as follows.

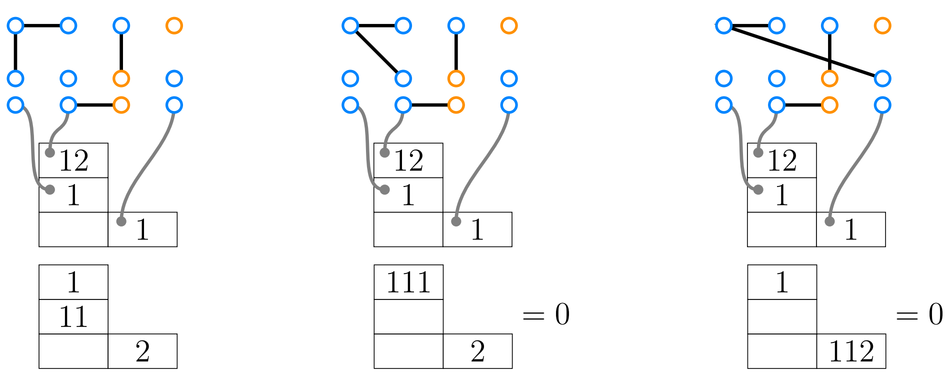
(i) Pull out the content of $\tilde{T}$, a multiset partition with r elements from $[k]$, in a row above and fix the order.

(ii) Place $\tilde{\pi}$ on top and permute the vertices of the same color at the bottom in each possible way.

(iii) For each permutation, compute the action as for P_r^λ.

(iv) Sum the resulting tableaux and divide by the number of permutations.

Example 3.3. The action of a multiset partition on a multiset partition tableau.



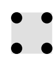
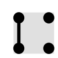
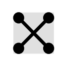
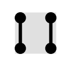
$$\tilde{\pi}.\begin{array}{|c|c}\cline{1-1} 12 \\ \cline{1-1} 1 \\ \hline & \multicolumn{1}{c|}{1} \\ \hline\end{array} = \frac{1}{3}\begin{array}{|c|c}\cline{1-1} 1 \\ \cline{1-1} 11 \\ \hline & \multicolumn{1}{c|}{2} \\ \hline\end{array} = \frac{-1}{3}\begin{array}{|c|c}\cline{1-1} 11 \\ \cline{1-1} 1 \\ \hline & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

Section 3.1

Characters

In the case of $P_r(n)$, each character value is some multiple n^c for $c \in \mathbb{Z}$ of the character for a standard diagram indexed by a partition $\lambda \vdash m \leq r$. Because of this fact, we can write a square table which captures all the character data of the algebra's representations. See [18, Section 2.2] for a beautiful algorithm for reducing an element of $P_r(n)$ to one of these standard diagrams to compute its character.

Example 3.4. Here we show the character table for $P_2(n)$ with the four standard diagrams across the top.

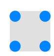
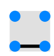
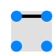
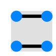
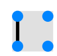
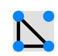
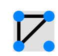
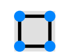
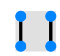
				
$\emptyset$	n^2	n	2	2
(1)	0	n	1	3
(2)	0	0	1	1
$(1,1)$	0	0	-1	1

Using Lemma 2.4, we have a straightforward averaging formula for the characters of the diagram-like basis elements acting on the irreducible representations. Write $\chi_A^\lambda(a)$ for the character of $a \in A$ acting on the irreducible representation A^λ.

Corollary 3.5. *For $\tilde{\pi} = \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi)$ with $\pi \in \Pi_{2r}$, its character on $\mathrm{MP}^\lambda_{r,k}$ can be computed* *as*

$$\chi^{\lambda}_{\mathrm{MP}_{r,k}(n)}(\mathcal{D}_{\tilde{\pi}}) = \delta_{\boldsymbol{a},\boldsymbol{b}} \frac{1}{|\mathfrak{S}_{\boldsymbol{a}}|} \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{a}}} \chi^{\lambda}_{P_r(n)}(\mathcal{L}_{\sigma.\pi}).$$

Example 3.6. Here we compute the characters for $\mathrm{MP}_{2,1}(n)$. Notice that unlike the partition algebra, not every character is a multiple of some standard character. It's not clear how to sort these diagrams into classes analogous to those for $P_2(n)$.

									
$\emptyset$	n^2	n	n	n	n	1	1	1	2
(1)	0	0	0	0	$\frac{1}{2}n$	1	1	1	2
(2)	0	0	0	0	0	0	0	0	1

While the above example points out the difficulty of producing a square character table for the multiset partition algebra, notice that we need only compute the character of an $\mathrm{MP}_{2,1}(n)$-module on three elements in the above table whose columns are linearly independent to have complete information on the character. The next step for the character theory of the multiset partition algebra is to determine such representatives in general.

Chapter 4

Generators

To describe a generating set for $\mathbb{MP}_{r,k}(n)$, we will give an algorithm for factoring out certain blocks from a diagram. We call a block of the form $\{\{i, \bar{i}\}\}$ a *vertical bar*. A block of a multiset partition $\tilde{\pi}$ which is not a vertical bar or a singleton is called a *nonbasic* block. We now define a statistic on multiset partitions and prove a lemma about how this statistic interacts with the diagram-like product. Write $N(\tilde{\pi})$ for the multiset of nonbasic blocks of $\tilde{\pi}$ and define the *nonbasic weight* of $\tilde{\pi}$ to be

$$\operatorname{nbw}(\tilde{\pi}) = \sum_{\tilde{B} \in N(\tilde{\pi})} \left| \tilde{B} \right|.$$

Lemma 4.1. *If $\mathcal{D}_{\tilde{\nu}}$ appears with nonzero coefficient in the product $\mathcal{D}_{\tilde{\pi}_1}\mathcal{D}_{\tilde{\pi}_2}$, then*

$$\operatorname{nbw}(\tilde{\nu}) \leq \operatorname{nbw}(\tilde{\pi}_1) + \operatorname{nbw}(\tilde{\pi}_2)$$

with equality if and only if $N(\tilde{\nu}) = N(\tilde{\pi}_1) \uplus N(\tilde{\pi}_2)$.

Proof. Consider a snapshot (π_1, π_2) in the product $\mathcal{D}_{\tilde{\pi}_1}\mathcal{D}_{\tilde{\pi}_2}$ and let $\nu = \pi_1 \circ \pi_2$. Suppose

$\tilde{\pi}_1 = \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi_1)$ and $\tilde{\pi}_2 = \kappa_{\boldsymbol{b},\boldsymbol{c}}(\pi_2)$. Write $\tilde{\nu} = \kappa_{\boldsymbol{a},\boldsymbol{c}}(\nu)$. We call a vertex of π_1, π_2, or ν nonbasic if it is mapped to an element of a nonbasic block under the coloring map. Our goal is now to construct an injective map φ from the set of nonbasic vertices of ν to the nonbasic vertices of π_1 and π_2.

Consider a nonbasic vertex v of ν labeled i. There is a corresponding vertex v' of π_1 also labeled i. If v' were the only element in its block, the same would be true for v. Hence, v' must either be a nonbasic vertex or one end of a vertical bar. In the first case, set $\varphi(v) = v'$. In the latter case, let $\bar{j}$ be the label of the other vertex in the vertical bar and let $\varphi(v)$ be the vertex of π_2 labeled j. If the diagram of π_1 were set atop that of π_2, this would be the vertex on the top of π_2 that the vertical bar lands on. If v is labeled $\bar{i}$, the same process is followed swapping which elements are barred (see Figure 4.1 for an illustration of this map).

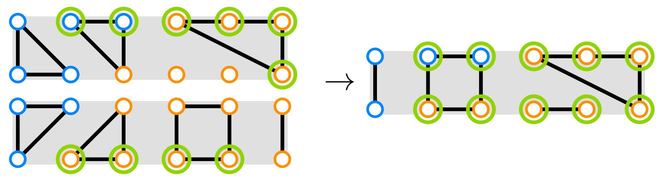


Figure 4.1: Illustration of the injective map φ constructed in Lemma 4.1. The nonbasic vertices on the right and the corresponding nonbasic vertices in the image of φ on the left are highlighted in green.

In the first case, it is clear that $\varphi(v)$ is nonbasic. In the latter case, if $\varphi(v)$ were basic, it would be either the only element of its block (in which case, v would be a singleton) or part of a vertical bar (in which case, v would be in a vertical bar). Either way, this contradicts the assumption that ν is nonbasic, so we have constructed a map φ from the set of nonbasic vertices of ν to the nonbasic vertices of π_1 and π_2. It's clear that this map is injective, and so the number of nonbasic vertices of ν is less than the total number of nonbasic vertices of π_1 and π_2, giving us the desired inequality.

Now to investigate the case of equality we consider how the map φ interacts with the set partition structure. Suppose that $\varphi(v)$ and $\varphi(w)$ are in the same block. Without loss of generality, assume they are in the same block of π_1. We then need to consider the following cases: (see Figure 4.2(i)-(iii) for illustrations of these cases)

(i) $\varphi(v)$ and $\varphi(w)$ are both on the top of the block.

The vertex $\varphi(v)$ has the same label as v and $\varphi(w)$ has the same label as w. Because the vertices with these labels are connected, the vertices with the same labels must be connected in the product, so v and w are in the same block.

(ii) $\varphi(v)$ and $\varphi(w)$ are both on the bottom of the block.

The vertices $\varphi(v)$ and $\varphi(w)$ each meet a vertical bar whose other end is labeled the same as v and w respectively. Hence, v and w are joined in the product.

(iii) Without loss of generality $\varphi(v)$ is on the top of the block and $\varphi(w)$ is on the bottom.

The vertex $\varphi(v)$ is labeled the same as v and the vertex on the other end of the vertical bar meeting $\varphi(w)$ is labeled the same as w. Hence v and w are again joined in the product.

Hence, if $\varphi(v)$ and $\varphi(w)$ are in the same block, then u and v are in the same block.

Suppose that v and w are in the same block but $\varphi(v)$ and $\varphi(w)$ are not (see Figure 4.2(iv)-(v)). Then two nonbasic blocks in the product must have been combined, and any vertex where the two nonbasic blocks meet must not be in the image of φ, meaning φ is not a surjection in this case.

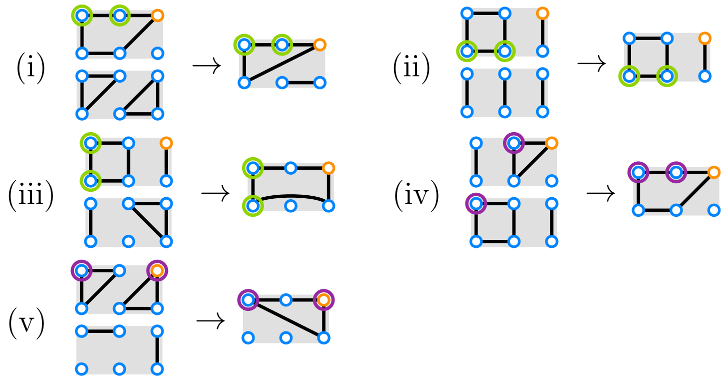
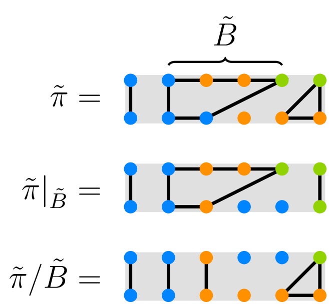


Figure 4.2: Illustrations of cases when $\varphi(u)$ and $\varphi(w)$ are in the same block.

When equality holds, the map φ is a bijection and $\varphi(v)$ in the same block as $\varphi(w)$ if and only if v and w are in the same block. The map φ then induces a bijection of nonbasic *blocks*, hence $N(\tilde{\nu}) = N(\tilde{\pi}_1) \uplus N(\tilde{\pi}_2)$. $\square$

We will introduce a sort of factorization of a diagram $\tilde{\pi}$ with a nonbasic block $\tilde{B}$ into diagrams with fewer nonbasic blocks, and to that end we define two diagrams $\tilde{\pi}/\tilde{B}$ and $\tilde{\pi}|_{\tilde{B}}$. Informally, the diagram $\tilde{\pi}/\tilde{B}$ is the result of removing the block $\tilde{B}$ and replacing it with basic blocks, and the diagram $\tilde{\pi}|_{\tilde{B}}$ is a diagram whose only nonbasic block is $\tilde{B}$.

Example 4.2. Here we show how the diagram $\tilde{\pi}$ can be factored at the nonbasic block $\tilde{B}$.



More precisely, let $\tilde{\pi} \in \tilde{\Pi}_{2r,k}$ have a nonbasic block $\tilde{B}$. Without loss of generality, assume $\tilde{B}$ has more unbarred entries than barred entries. Write $\tilde{\pi}/\tilde{B}$ for the multiset partition obtained by replacing $\tilde{B}$ with vertical bars $\{\{i, \bar{i}\}\}$ for each barred entry $\bar{i}$ of $\tilde{B}$ and a number of singletons $\{\{\bar{1}\}\}$ making up the difference in the number of barred and unbarred entries in $\tilde{B}$. Write $\tilde{\pi}|_{\tilde{B}}$ for the multiset partition consisting of $\tilde{B}$, a vertical bar $\{\{i, \bar{i}\}\}$ for each vertex labeled i in $\tilde{\pi}$ not in $\tilde{B}$, and enough $\{\{\bar{1}\}\}$ to make it an element of $\tilde{\Pi}_{2r,k}$.

We see immediately in Example 4.2 that although the element $\mathcal{D}_{\tilde{\pi}}$ will appear with nonzero coefficient in $\mathcal{D}_{\tilde{\pi}|_{\tilde{B}}}\mathcal{D}_{\tilde{\pi}/\tilde{B}}$, many other diagrams appear. We now define a partial order on the multiset partition diagrams with respect to which these extra diagrams are smaller than $\tilde{\pi}$. This will allow us to use the factorization recursively to write $\mathcal{D}_{\tilde{\pi}}$ as a polynomial in simpler diagrams.

Write $\mathrm{vb}(\tilde{\pi})$ for the number of vertical bars in $\tilde{\pi}$. Define a partial order on $\tilde{\Pi}_{r,k}$ by saying that $\tilde{\pi} \prec \tilde{\tau}$ if either

$$\mathrm{nbw}(\tilde{\pi}) < \mathrm{nbw}(\tilde{\tau})$$

or

$$\mathrm{nbw}(\tilde{\pi}) = \mathrm{nbw}(\tilde{\tau}) \text{ and } \mathrm{vb}(\tilde{\pi}) < \mathrm{vb}(\tilde{\tau}).$$

Lemma 4.3. *Let $\tilde{\pi}$ be a multiset partition and $\tilde{B}$ a nonbasic block of $\tilde{\pi}$ which has more unbarred entries than barred entries. Then there is a constant $c \in \mathbb{C}$ so that*

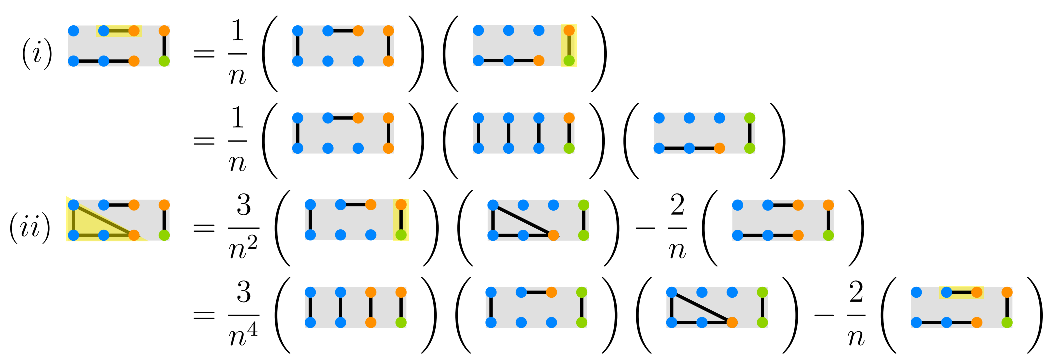
$$c\mathcal{D}_{\tilde{\pi}|_{\tilde{B}}}\mathcal{D}_{\tilde{\pi}/\tilde{B}} - \mathcal{D}_{\tilde{\pi}} \in \mathrm{span}_{\mathbb{C}}\{\mathcal{D}_{\tilde{\nu}} : \tilde{\nu} \prec \tilde{\pi}\}.$$

Proof. Consider a snapshot in the product $\mathcal{D}_{\tilde{\pi}|_{\tilde{B}}}\mathcal{D}_{\tilde{\pi}/\tilde{B}}$ in which each vertex at the bottom of the block $\tilde{B}$ in $\tilde{\pi}|_{\tilde{B}}$ meets a vertical bar in $\tilde{\pi}/\tilde{B}$ and each singleton in $\tilde{\pi}|_{\tilde{B}}$ meets a singleton in $\tilde{\pi}/\tilde{B}$. The resulting diagram from this snapshot is $\tilde{\pi}$, and so $\mathcal{D}_{\tilde{\pi}}$ appears with nonzero coefficient in the product. Let c be the reciprocal of the coefficient it appears with.

By Lemma 4.1 any $\mathcal{D}_{\tilde{\nu}}$ appearing in the product must have $\mathrm{nbw}(\tilde{\nu}) \leq \mathrm{nbw}(\tilde{\pi}|_{\tilde{B}}) + \mathrm{nbw}(\tilde{\pi}/\tilde{B}) = \mathrm{nbw}(\tilde{\pi})$ with equality only if $N(\tilde{\nu}) = N(\tilde{\pi}|_{\tilde{B}}) \biguplus N(\tilde{\pi}/\tilde{B}) = N(\tilde{\pi})$.

If any vertical bars in $\tilde{\nu}$ came from nonbasic blocks of $\tilde{\pi}|_{\tilde{B}}$ and $\tilde{\pi}/\tilde{B}$ meeting, then $\tilde{\nu}$ would necessarily have a smaller nonbasic weight. Hence, in the case that $\mathrm{nbw}(\tilde{\nu}) = \mathrm{nbw}(\tilde{\pi})$, it must either be the case that $\tilde{\nu} = \tilde{\pi}$ or $\tilde{\nu}$ has fewer vertical bars. So, every $\tilde{\nu} \neq \tilde{\pi}$ that appears in the product is smaller in $(\tilde{\Pi}_{2r,k}, \preceq)$. $\square$

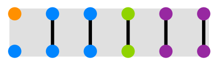
Example 4.4. Lemma 4.3 can be used recursively to write a diagram as a polynomial in diagrams with a single nonbasic block. At each step, the nonbasic block $\tilde{B}$ that the diagram is being factored at is highlighted. Notice that example (ii) ends where example (i) begins.

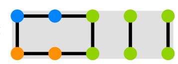


We now introduce our generators. For $i, j \in [k]$ and $\boldsymbol{a} \in W_{r-1,k}$, write $P_{i,j,\boldsymbol{a}}$ for the diagram-like basis element indexed by the set partition with singleton blocks $\{\{i\}\}$ and $\{\{\bar{j}\}\}$ and vertical bars whose colors have multiplicity given by $\boldsymbol{a}$. Now fix $i \in [r]$,

$\boldsymbol{a}, \boldsymbol{b} \in W_{i,k}$ and $\boldsymbol{c} \in W_{r-i,k}$. Write $R_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$ for the diagram-like basis element for the set partition with a block whose colors on top and bottom are given by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively as well as vertical bars with multiplicities given by $\boldsymbol{c}$.

Example 4.5. Here we show an example of each type of generator.

$P_{2,1,(2,0,1,2)} =$  $\in \mathrm{MP}_{6,4}(x)$

$R_{(2,0,1),(0,2,1),(0,0,2)} =$  $\in \mathrm{MP}_{5,3}(x)$

As a base case, we show how these elements generate the diagrams with no nonbasic blocks.

Lemma 4.6. *The elements* $\{P_{i,j,\boldsymbol{a}} : i, j \in [k], \boldsymbol{a} \in W_{r-1,k}\}$ *generate each* $\mathcal{D}_{\tilde{\pi}}$ *where* $\tilde{\pi}$ *has no nonbasic blocks.*

Proof. Fix $\boldsymbol{b} \in W_{r,k}$. For $m \leq \boldsymbol{b}_1$, write Q_m for the diagram-like basis element indexed by the multiset partition with m pairs of singletons $\{\!\{1\}\!\}$ and $\{\!\{\bar{1}\}\!\}$ along with vertical bars $\{\!\{1, \bar{1}\}\!\}^{\boldsymbol{b}_1 - m}, \{\!\{2, \bar{2}\}\!\}^{\boldsymbol{b}_2}, \ldots, \{\!\{k, \bar{k}\}\!\}^{\boldsymbol{b}_k}$. Notice that $Q_1 = P_{1,1,\boldsymbol{b}'}$ where $\boldsymbol{b}'$ is $\boldsymbol{b}$ with the first entry decremented by one.

Now consider the product $Q_1 Q_m$. The singleton at the bottom of Q_1 will meet one of the m singletons at the top of Q_m in $\frac{m}{\boldsymbol{b}_1}$ of the snapshots. In the remaining snapshots, the singleton meets a vertical bar and breaks it into a singleton, resulting in Q_{m+1}. This observation results in the formula

$$Q_1 Q_m = \frac{m}{\boldsymbol{b}_1} n Q_m + \frac{\boldsymbol{b}_1 - m}{\boldsymbol{b}_1} Q_{m+1}$$

for the product. Hence, the elements Q_m for $1 \leq m \leq \boldsymbol{b}_1$ are generated by the elements $P_{i,j,\boldsymbol{a}}$ (see Figure 4.3(i)).

Suppose $\tilde{\pi}$ is a multiset partition with no nonbasic blocks and a singleton $\{\!\{i\}\!\}$ with $i \neq 1$. Let $\tilde{\pi}'$ be the result of replacing that $\{\!\{i\}\!\}$ with $\{\!\{1\}\!\}$. Then for $\boldsymbol{c} \in W_{r-1,k}$ chosen so that $P_{i,1,\boldsymbol{c}}\mathcal{D}_{\tilde{\pi}'}$ is nonzero, this product includes $\mathcal{D}_{\tilde{\pi}}$ along with diagrams with fewer vertical bars (see Figure 4.3(ii)). Via this process and the corresponding process for singletons $\{\!\{\bar{i}\}\!\}$, we can write any basic diagram with a non-one singleton as a polynomial in diagrams with fewer non-one singletons or fewer vertical bars. Repeating this process for any diagram in the resulting polynomial with a non-one singleton terminates in a polynomial in diagrams with all basic blocks and singletons of the form $\{\!\{1\}\!\}$ or $\{\!\{\bar{1}\}\!\}$. These are just the Q_m above for different choices of $\boldsymbol{b}$, so the $\{P_{i,j,\boldsymbol{a}} : i, j \in [k], \boldsymbol{a} \in W_{r-1,k}\}$ generate the diagrams with all basic blocks. $\square$

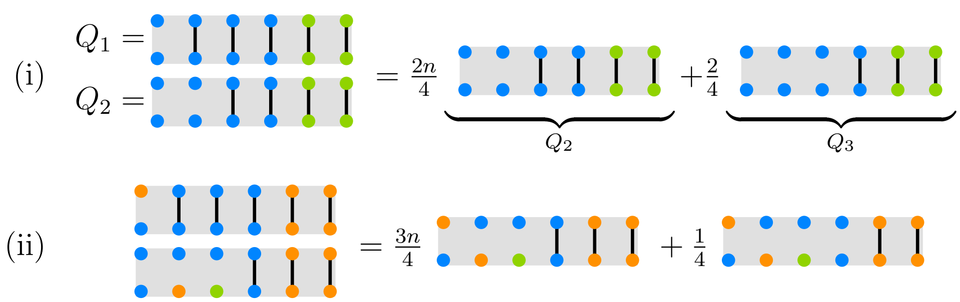


Figure 4.3: Examples of the processes employed in Lemma 4.6.

Now we want to understand how we can build up diagrams which do have some nonbasic blocks.

Theorem 4.7. *The algebra* $\mathrm{MP}_{r,k}(x)$ *is generated by the set*

$$\Theta = \{P_{i,j,\boldsymbol{a}} : i, j \in [k], \boldsymbol{a} \in W_{r-1,k}\} \cup \{R_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} : \boldsymbol{a}, \boldsymbol{b} \in W_{i,k}, \boldsymbol{c} \in W_{r-i,k} \textit{ for some } i \in [r]\}.$$

Proof. If $\tilde{\pi}$ has more than one nonbasic block, then we apply Lemma 4.3 to write $\mathcal{D}_{\tilde{\pi}}$ as a polynomial in elements $\mathcal{D}_{\tilde{\nu}}$ where $\tilde{\nu} \prec \tilde{\pi}$. We can iterate this process on each $\mathcal{D}_{\tilde{\nu}}$ in this polynomial where $\tilde{\nu}$ has more than one nonbasic block (see Example 4.4). Because the poset $(\tilde{\Pi}_{2r,k}, \preceq)$ is finite, this iteration terminates with $\mathcal{D}_{\tilde{\pi}}$ written as a polynomial in elements $\mathcal{D}_{\tilde{\nu}}$ where each $\tilde{\nu}$ has at most one nonbasic block. Hence, it suffices to show that Θ generates the elements $\mathcal{D}_{\tilde{\nu}}$ where $\tilde{\nu}$ has at most one nonbasic block.

By Lemma 4.6, Θ generates the elements $\mathcal{D}_{\tilde{\pi}}$ where $\tilde{\pi}$ has no nonbasic blocks. We now prove that Θ generates the diagrams with a single nonbasic block by induction on the number of vertical bars. If $\tilde{\pi}$ has a single nonbasic block and no vertical bars, it has a straightforward factorization as $\mathcal{D}_{\tilde{\pi}} = \mathcal{D}_{\tilde{\pi}_1}\mathcal{D}_{\tilde{\pi}_2}\mathcal{D}_{\tilde{\pi}_3}$ where $\tilde{\pi}_2$ is obtained from $\tilde{\pi}$ by connecting all vertices into a single block, $\tilde{\pi}_1$ has a vertical bar for each vertex at the top of the nonbasic block of $\tilde{\pi}$ and a pair of identically colored singletons for each singleton atop $\tilde{\pi}$, and $\tilde{\pi}_3$ is obtained similarly from the bottom of $\tilde{\pi}$ (see Figure 4.4(i)). Notice that $\mathcal{D}_{\tilde{\pi}_1}, \mathcal{D}_{\tilde{\pi}_2}, \mathcal{D}_{\tilde{\pi}_3} \in \Theta$.

For $\tilde{\pi}$ with a single nonbasic block and s vertical bars, one can try a modified version of the above factorization in which a copy of each vertical bar in $\tilde{\pi}$ is put in $\tilde{\pi}_1$, $\tilde{\pi}_2$, and $\tilde{\pi}_3$ (see Figure 4.4(ii)). The element $\mathcal{D}_{\tilde{\pi}}$ appears in the product $\mathcal{D}_{\tilde{\pi}_1}\mathcal{D}_{\tilde{\pi}_2}\mathcal{D}_{\tilde{\pi}_3}$ when each singleton in $\tilde{\pi}_1$ and $\tilde{\pi}_3$ meets the nonbasic block in $\tilde{\pi}_2$. When these singletons instead meet vertical bars in $\tilde{\pi}_2$, the resulting diagram has fewer than s vertical bars. By induction on the number of vertical bars, the set Θ generates the diagrams with at most one nonbasic block and hence the algebra $\mathbb{MP}_{r,k}(x)$. $\square$

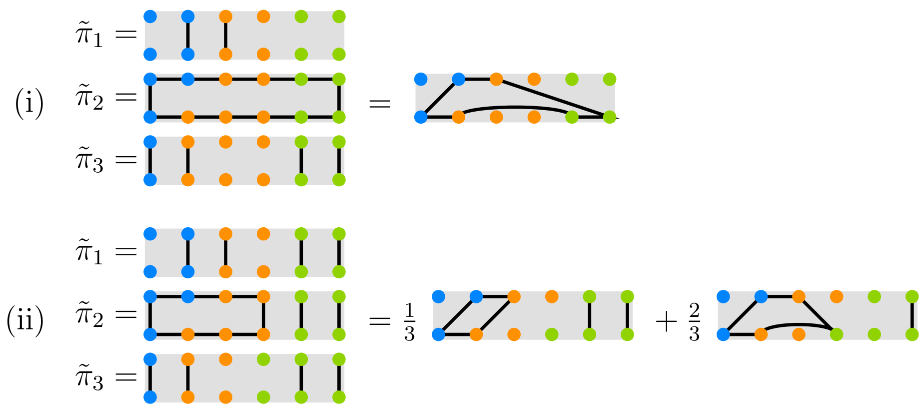


Figure 4.4: Examples of the products in Theorem 4.7.

Chapter 5

Applications to Representations of $\mathfrak{S}_n$ and GL_n

Up to this point we have used the double centralizer theorem largely in one direction: to leverage information about the representation theory of the symmetric group to understand the indexing sets and dimensions of irreducible $\mathbb{MP}_{r,k}(n)$ modules, but we have mentioned that information about representation theory can flow both ways across these dualities. In this section, we restrict our attention to an object called the balanced multiset partition algebra and use it to highlight how questions about the representation theory of $\mathfrak{S}_n$ and GL_n can be cast in a new light.

Section 5.1

The Balanced Multiset Partition Algebra

The *symmetric power* $\mathrm{Sym}^r(V_n)$ of V_n is the quotient of the tensor power ${V_n}^{\otimes r}$ by the $\mathfrak{S}_r$-action. For a weak composition $\boldsymbol{a} \in W_{r,k}$, we generalize this notation to

$$\mathrm{Sym}^{\boldsymbol{a}}(V_n) \cong \mathrm{Sym}^{\boldsymbol{a}_1}(V_n) \otimes \cdots \otimes \mathrm{Sym}^{\boldsymbol{a}_k}(V_n) \cong s_{\boldsymbol{a}} {V_n}^{\otimes r}.$$

The *balanced multiset partition algebra* $\mathbb{MP}_{\boldsymbol{a}}(n) \cong \mathrm{End}_{\mathfrak{S}_n}(\mathrm{Sym}^{\boldsymbol{a}}(V_n))$ was introduced in [37] by Narayanan, Paul, and Srivastava (although they called it the multiset partition algebra—we add the extra adjective to disambiguate from the algebra introduced by Orellana and Zabrocki). By Lemma 2.1, we can write

$$\begin{aligned}
\mathbb{MP}_{\boldsymbol{a}}(n) &\cong \mathrm{End}_{\mathfrak{S}_n}(\mathrm{Sym}^{\boldsymbol{a}}(V_n)) \\
&\cong \mathrm{End}_{\mathfrak{S}_n}(s_{\boldsymbol{a}} {V_n}^{\otimes r}) \\
&\cong s_{\boldsymbol{a}} \,\mathrm{End}_{\mathfrak{S}_n}({V_n}^{\otimes r}) s_{\boldsymbol{a}} \\
&\cong s_{\boldsymbol{a}} P_r(n) s_{\boldsymbol{a}}
\end{aligned}$$

for $n \geq 2r$. Hence, we can think of $\mathbb{MP}_{\boldsymbol{a}}(n)$ as the subspace of diagrams in $\mathbb{MP}_{r,k}(n)$ whose multiplicites of colors on top and bottom are both $\boldsymbol{a}$. The only thing keeping it from being a sub*algebra* is the fact that its identity element is not the same as that of $\mathbb{MP}_{r,k}(n)$.

The methods of Chapter 3 can be applied to constructing the irreducible representations of $\mathbb{MP}_{\boldsymbol{a}}(n)$. By Lemma 2.3, the set $\{s_{\boldsymbol{a}} P_r^{\lambda} : \lambda \in \Lambda^{P_r(n)}\}$ contains all the irreducible representations of $\mathbb{MP}_{\boldsymbol{a}}$ along with possibly the zero-module. Let

$\mathcal{SSMPT}(\lambda, \boldsymbol{a})$ be the set of semistandard multiset partition tableaux of shape λ where the number of is is given by $\boldsymbol{a}_i$ and let $\Lambda^{\mathbb{MP}_{\boldsymbol{a}}(n)}$ be the set of $\lambda \vdash n$ for which $\mathcal{SSMPT}(\lambda, \boldsymbol{a}) \neq \emptyset$. Define

$$\mathbb{MP}_{\boldsymbol{a}}^{\lambda} := s_{\boldsymbol{a}} P_r^{\lambda}.$$

By an argument analogous to that in Theorem 3.2 (where the RSK algorithm of Section 1.4 is restricted to the basis elements of $\mathbb{MP}_{\boldsymbol{a}}(n)$), we have for $n \geq 2r$ that $\{\mathbb{MP}_{\boldsymbol{a}}^{\lambda} : \lambda \in \Lambda^{\mathbb{MP}_{\boldsymbol{a}}(n)}\}$ forms a complete set of irreducible representations for $\mathbb{MP}_{\boldsymbol{a}}(n)$ and that the set $\{w_{\tilde{T}} : \tilde{T} \in \mathcal{SSMPT}(\lambda, \boldsymbol{a})\}$ forms a basis for $\mathbb{MP}_{\boldsymbol{a}}^{\lambda}$.

Section 5.2

The Kronecker Problem

Recall that the Kronecker coefficient $g_{\lambda\mu}^{\nu}$ is the multiplicity of the Specht module S^{ν} in the decomposition

$$S^{\lambda} \otimes S^{\mu} \cong \bigoplus_{\nu \vdash n} (S^{\nu})^{\oplus g_{\lambda\mu}^{\nu}}.$$

For a partition λ with $n - |\lambda| \geq \lambda_1$, write $\lambda[n]$ for the partition obtained by adding a first row to make it a partition of n. It is well-known that for sufficiently large n, the value $g_{\lambda[n]\mu[n]}^{\nu[n]}$ is eventually a constant $\overline{g}_{\lambda\mu}^{\nu}$ called the *reduced Kronecker coefficient* [9, 17, 35, 36].

There is a combinatorial interpretation when one of the indexing partitions is a hook shape (given by Blasiak [6]—see also [31]) and a positive formula when one of the indexing partitions is a two-part partition (given by Ballantine and Orellana [3]).

A unified approach to both of these cases was given by Bowman, De Visscher and Orellana [7] using the partition algebra to write the reduced Kronecker coefficients in these cases in terms of the Littlewood-Richardson coefficients. The key result for writing these formulas is that the Kronecker coefficient appears in the restriction of simple $P_r(n)$-modules to subalgebras analogous to Young subgroups:

Theorem 5.1 ([7, Corollary 3.4])**.**

$$\mathrm{Res}^{P_{r+s}(n)}_{P_r(n)\otimes P_s(n)}(P^{\nu}_{r+s}) \cong \bigoplus_{\lambda,\mu} \left(P^{\lambda}_{r} \otimes P^{\mu}_{r}\right)^{\oplus \overline{g}^{\nu}_{\lambda\mu}}$$

Like for the partition algebra, Kronecker coefficients appear in the restriction of simple $\mathbb{MP}_{\boldsymbol{a}}(n)$-modules to a particular subalgebra. Let $\mathbb{MP}'_{\boldsymbol{a}}(n)$, called the *Young-type subalgebra* of $\mathbb{MP}_{\boldsymbol{a}}(n)$, be the subalgebra in which no two vertices of different colors are connected by an edge. It's clear that

$$\mathbb{MP}'_{\boldsymbol{a}}(n) \cong \mathbb{MP}_{\boldsymbol{a}_1,1}(n) \otimes \cdots \otimes \mathbb{MP}_{\boldsymbol{a}_k,1}(n).$$

We want to restrict the $\mathbb{MP}_{\boldsymbol{a}}(n)$-module $\mathbb{MP}^{\lambda}_{\boldsymbol{a}}$ to this Young-type subalgebra. By the double centralizer theorem, $\mathbb{MP}^{\lambda}_{\boldsymbol{a}} \cong \mathrm{Hom}_{\mathfrak{S}_n}(S^{\lambda}, \mathrm{Sym}^{\boldsymbol{a}}(V_n))$, so we should consider how $\mathrm{Sym}^{\boldsymbol{a}}(V_n)$ decomposes as an $\mathbb{MP}'_{\boldsymbol{a}}(n) \times \mathfrak{S}_n$-module.

$$
\begin{aligned}
\mathrm{Sym}^{\boldsymbol{a}}(V_n) &\cong \mathrm{Sym}^{\boldsymbol{a}_1}(V_n)\otimes\cdots\otimes\mathrm{Sym}^{\boldsymbol{a}_k}(V_n)\\
&\cong \bigotimes_{i=1}^{k}\bigoplus_{\nu^i\vdash n}\mathrm{MP}^{\nu^i}_{\boldsymbol{a}_i,1}\otimes S^{\nu_i}\\
&\cong \bigoplus_{\nu^1,\ldots,\nu^k\vdash n}\mathrm{MP}^{\nu^1}_{\boldsymbol{a},1}\otimes\cdots\otimes\mathrm{MP}^{\nu^k}_{\boldsymbol{a}_k,1}\otimes S^{\nu^1}\otimes\cdots\otimes S^{\nu^k}\\
\mathrm{MP}^{\lambda}_{\boldsymbol{a}} &\cong \mathrm{Hom}_{\mathfrak{S}_n}(S^{\lambda},\mathrm{Sym}^{\boldsymbol{a}}(V_n))\\
&\cong \bigoplus_{\nu^1,\ldots,\nu^k}(\mathrm{MP}^{\nu^1}_{\boldsymbol{a}_1,1}\otimes\cdots\otimes\mathrm{MP}^{\nu^k}_{\boldsymbol{a}_k,1})\otimes\mathrm{Hom}_{\mathfrak{S}_n}(S^{\lambda},S^{\nu^1}\otimes\cdots\otimes S^{\nu^k})
\end{aligned}
$$

The above observations immediately leads to the following theorem relating restrictions to the Young-type subalgebras with the Kronecker problem, which analogizes [41, Theorem 5.10].

Theorem 5.2. *For $\boldsymbol{a}\in W_{r,k}$, $n\geq 2r$, and $\nu\in\Lambda^{\mathrm{MP}_{\boldsymbol{a}}(n)}$,*

$$
\mathrm{Res}^{\mathrm{MP}_{\boldsymbol{a}}(n)}_{\mathrm{MP}'_{\boldsymbol{a}}(n)}(\mathrm{MP}^{\nu}_{\boldsymbol{a}})\cong\bigoplus_{\lambda^1,\ldots,\lambda^k\vdash n}\left(\mathrm{MP}^{\lambda^1}_{\boldsymbol{a}_1,1}\otimes\cdots\otimes\mathrm{MP}^{\lambda^k}_{\boldsymbol{a}_k,1}\right)^{g^{\nu}_{\lambda^1,\ldots,\lambda^k}}
$$

where $g^{\nu}_{\lambda^1,\ldots,\lambda^k}$ is the multiplicity of S^{ν} in $S^{\lambda^1}\otimes\cdots\otimes S^{\lambda^k}$.

Section 5.3

The Restriction Problem

The restriction problem asks how a simple GL_n module restricts to the subgroup of $n\times n$ permutation matrices. Such a restriction decomposes as follows into simple

$\mathfrak{S}_n$-modules:

$$\operatorname{Res}^{GL_n}_{\mathfrak{S}_n}(GL_n^\lambda) \cong \bigoplus_{\mu : n-|\mu| \geq \mu_1} \left(S^{\mu[n]}\right)^{\oplus r_{\lambda\mu}(n)}.$$

The multiplicity $r_{\lambda\mu}(n)$ is called the *restriction coefficient*, and it is well-known that this coefficient is eventually constant for sufficiently large, a result attributed to D. E. Littlewood in [2]. We write $r_{\lambda\mu}$ for its eventual constant value and call this the *stable restriction coefficient*. The restriction problem asks for a combinatorial interpretation of $r_{\lambda\mu}$.

Orellana and Zabrocki [40] (and independently Assaf and Speyer [2]) introduced a basis $\tilde{s}_\lambda$ of symmetric functions which encodes the restriction coefficients. In particular, the restriction coefficients give the change of basis from the Schur functions s_λ to the basis $\tilde{s}_\lambda$.

Two cases of the restriction problem are particularly well-understood. First, when $\lambda = (k)$, we are restricting the symmetric power $\mathrm{Sym}^{(k)}$. This case is studied in [1, 23, 38]. Second, when $\lambda = (1^k)$, we are restricting the exterior power. This case is studied in [1, 38]. See also exercises 7.72 and 7.73 in [49].

In this section, we first show how a combinatorial interpretation for the restriction coefficients in the case of the symmetric power and exterior power can be straightforwardly recovered from the objects and methods in this thesis. Then, we conclude with a discussion of how these methods could be applied to further cases of the restriction problem.

5.3.1. Restriction of Symmetric Powers

We can use the irreducible $\mathbb{MP}_{\boldsymbol{a}}(n)$-modules to study the restriction problem for the GL_n-module $\mathrm{Sym}^{\boldsymbol{a}}(V_n)$ for $n \geq |\boldsymbol{a}|$. The double centralizer theorem tells us that as an $\mathfrak{S}_n \times \mathbb{MP}_{\boldsymbol{a}}(n)$-module,

$$\mathrm{Sym}^{\boldsymbol{a}}(V_n) \cong \bigoplus_{\lambda \in \Lambda^{\mathbb{MP}_{\boldsymbol{a}}(n)}} S^\lambda \otimes W^\lambda_{\mathbb{MP}_{\boldsymbol{a}}(n)}$$

where

$$\begin{aligned} W^\lambda_{\mathbb{MP}_{\boldsymbol{a}}(n)} &= \mathrm{Hom}(S^\lambda, \mathrm{Sym}^{\boldsymbol{a}}(V_n)) \\ &\cong \mathrm{Hom}(S^\lambda, s_{\boldsymbol{a}} {V_n}^{\otimes r}) \\ &\cong s_{\boldsymbol{a}} \mathrm{Hom}(S^\lambda, {V_n}^{\otimes r}) \\ &\cong s_{\boldsymbol{a}} P^\lambda_r \\ &\cong \mathbb{MP}^\lambda_{\boldsymbol{a}} \end{aligned}$$

Then the multiplicity of S^λ in $\mathrm{Sym}^{\boldsymbol{a}}(V_n)$ as an $\mathfrak{S}_n$-module is given by the dimension of $\mathbb{MP}^\lambda_{\boldsymbol{a}}$. This allows us to give an alternative proof of a theorem of Harman [23] and of Orellana and Zabrocki [40].

Theorem 5.3 ([23, Proposition 3.11][40, Theorem 9])**.** *For $n \geq 2\,|\boldsymbol{a}|$,*

$$\mathrm{Res}^{GL_n}_{\mathfrak{S}_n}(\mathrm{Sym}^{\boldsymbol{a}}(V_n)) \cong \bigoplus_{\lambda \in \Lambda^{\mathbb{MP}_{\boldsymbol{a}}(n)}} (S^\lambda)^{\oplus d_{\boldsymbol{a},\lambda}}$$

where $d_{\boldsymbol{a},\lambda} = \dim(\mathbb{MP}^\lambda_{\boldsymbol{a}}) = |\mathcal{SSMPT}(\lambda, \boldsymbol{a})|$.

5.3.2. Restriction of Exterior Powers

We can apply similar principals to the restriction problem for the exterior power $GL_n^{(1^r)} = a_{(1^r)}V_n^{\otimes r}$ (in the notation of Section 1.2.4). By Lemma 2.1, we can write

$$\begin{aligned}\mathrm{End}_{\mathfrak{S}_n}(GL_n^{(1^r)}) &= \mathrm{End}_{\mathfrak{S}_n}(a_{(1^r)}V_n^{\otimes r}) \\ &\cong a_{(1^r)}P_r(n)a_{(1^r)}.\end{aligned}$$

Write $A_r(n) = a_{(1^r)}P_r(n)a_{(1^r)}$.

Then by the double centralizer theorem, we can decompose the exterior power as an $\mathfrak{S}_n \times A(n)$-module:

$$GL_n^{(1^r)} \cong \bigoplus_{\lambda \in \Lambda^{A_r(n)}} \mathfrak{S}^\lambda \otimes A_r^\lambda$$

where $A^\lambda = a_{(1^r)}P_r^\lambda$ is the simple $A_r(n)$-module indexed by λ. This means that the restriction coefficient $r_{(1^r),\mu}(n) = \dim(A_r^{\mu[n]})$ whenever $n \geq 2r$.

The structure of the algebra $A_r(n)$ is very simple. For $\pi \in \Pi_{2r}$, write $\mathcal{E}_\pi = a_{(1^r)}\mathcal{L}_\pi a_{(1^r)}$. It's clear that if π' is obtained by π by swapping two unbarred or two barred values, then $\mathcal{E}_{\pi'} = -\mathcal{E}_\pi$. Hence, $\mathcal{E}_\pi = 0$ if π has any block with more than one barred or unbarred value and if π has any two singleton blocks on the top or bottom, so the set $\{\mathcal{E}_\pi, \mathcal{E}_\tau\}$ forms a basis for $A_r(n)$ where π is the set partition corresponding to the identity permutation and τ is the set partition obtained from π by disconnecting one block. It's clear that $A_r(n)$ is a commutative algebra, so it must have two one-dimensional simple modules. We then need only determine for what values λ that $a_{(1^r)}P_r^\lambda$ is nonzero.

Lemma 5.4. *The module $a_{(1^r)}P_r^\lambda$ is nonzero if and only if $\lambda = (n-r+1, 1^{r-1})$ or $\lambda = (n-r, 1^r)$.*

Proof. For $T \in \mathcal{SSPT}(\lambda, r)$, let $y_T = a_{(1^r)}v_T$. As above, if T' is obtained from T by swapping two values, $y_{T'} = -y_T$, and so $y_T = 0$ if T has any box which contains more than one number or multiple non-empty boxes in the first row. Now supppose $\lambda = (m, 1^h)$ is a hook shape with $h = r$ or $h = r-1$ and let T be the set partition tableau with at most one number in each box where the numbers $1, \ldots, h$ are in the first column and the remaining number (if $h = r-1$) in the first row. Let $\{\tau_i : 1 \leq i \leq k\}$ be a transversal of $\mathfrak{S}_h$ in $\mathfrak{S}_r$. Then

$$\begin{aligned} y_T &= \frac{1}{r!}\sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma)\sigma v_T \\ &= \frac{1}{r!}\sum_{i=1}^{k}\sum_{\sigma' \in \mathfrak{S}_h} \operatorname{sgn}(\tau_i\sigma')\tau_i\sigma' v_T \\ &= \frac{1}{r!}\sum_{i=1}^{k}\sum_{\sigma' \in \mathfrak{S}_h} \operatorname{sgn}(\tau_i\sigma')\tau_i \operatorname{sgn}(\sigma') v_T \\ &= \frac{h!}{r!}\sum_{i=1}^{k} \operatorname{sgn}(\tau_i)\tau_i v_T. \end{aligned}$$

But if $k > 1$, then each $\tau_i v_T$ has a different set of numbers in the boxes above the first row, so there can be no cancellation between the terms. Hence $y_T \neq 0$.

Conversely, if λ is not a hook shape, it has two adjacent boxes in the same row above the first row. Let $T \in \mathcal{SSPT}(\lambda, r)$ and let $g_{A,B}$ be the Garnir element associated to these two boxes in T. Let $\{\nu_i : i \leq 1 \leq k\}$ be a transversal for $\mathfrak{S}_{A\cup B}$ in $\mathfrak{S}_r$.

Then

$$a_{(1^r)} = \frac{1}{r!}\left(\sum_{i=1}^{k} \operatorname{sgn}(\nu_i)\nu_i\right) g_{A,B} \left(\sum_{\sigma\in\mathfrak{S}_A\times\mathfrak{S}_B} \operatorname{sgn}(\sigma)\sigma\right)$$

and so because each σ only permutes the contents of boxes in the same column,

$$\begin{aligned}
y_T &= \frac{1}{r!}\left(\sum_{i=1}^{k} \operatorname{sgn}(\nu_i)\nu_i\right) g_{A,B} \sum_{\sigma\in\mathfrak{S}_A\times\mathfrak{S}_B} \operatorname{sgn}(\sigma)\sigma v_T \\
&= \frac{1}{r!}\left(\sum_{i=1}^{k} \operatorname{sgn}(\nu_i)\nu_i\right) g_{A,B} \sum_{\sigma\in\mathfrak{S}_A\times\mathfrak{S}_B} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma) v_T \\
&= \frac{|\mathfrak{S}_A\times\mathfrak{S}_B|}{r!}\left(\sum_{i=1}^{k} \operatorname{sgn}(\nu_i)\nu_i\right) g_{A,B} v_T \\
&= \frac{|\mathfrak{S}_A\times\mathfrak{S}_B|}{r!}\left(\sum_{i=1}^{k} \operatorname{sgn}(\nu_i)\nu_i\right) 0 \\
&= 0
\end{aligned}$$

□

Now that we've characterized the dimensions of the simple $A_r(n)$-modules, we can state the following theorem, which is a special case of [39, Theorem 5.1].

Theorem 5.5. *For $r > 0$ and $n \geq 2r$,*

$$r_{(1^r),\mu}(n) = \begin{cases} 1 & \textit{if } \mu = (1^r) \textit{ or } \mu = (1^{r-1}) \\ 0 & \textit{else} \end{cases}$$

5.3.3. Further Applications to the Restriction Problem

We can use the previous two sections as a model to attack the restriction problem for general $GL_n^\lambda = a_\lambda {V_n}^{\otimes r}$. Let $A_\lambda(n) = a_\lambda P_r(n) a_\lambda$ for $n \geq 2\,|\lambda|$. Then the multiplicity of $\mathfrak{S}^{\mu[n]}$ in GL_n^λ is the dimension of the simple module $A_\lambda^{\mu[n]} = a_\lambda P_r^{\mu[n]}$. Addressing linear independence in $a_\lambda P_r^{\mu[n]}$ directly is difficult, so the program suggested by the above two sections is as follows.

(a) Interpret the elements $a_\lambda L_\pi a_\lambda$ diagramatically to find a basis for $A_\lambda(n)$.

(b) Conjecture what tableaux the elements $a_\lambda v_T \in a_\lambda P_r^{\mu[n]}$ correspond to.

(c) Use a modified RSK algorithm to prove a basis for $A_\lambda^{\mu[n]}$.

We end this section with a straightforward enumerative result which supports the usefulness of multiset partition tableaux in studying the restriction problem. For $n \geq 2\,|\lambda|$,

$$r_{\lambda,\mu}(n) = \dim(a_\lambda P_r^{\mu[n]}) \leq \dim(r_\lambda P_r^{\mu[n]}) = |\mathcal{SSMPT}(\mu[n], \lambda)|$$

Hence, the restriction coefficients could be enumerated by multiset partition tableaux of shape $\mu[n]$ with content given by λ with some conditions.

Appendix A

Change of Basis

In this appendix, we give a formula for the change-of-basis from Orellana and Zabrocki's orbit basis (Section 1.3.2) to the diagram-like basis (Section 2.3). While the earlier sections dealt with the centralizer algebras $P_r(n)$ and $\mathbb{MP}_{r,k}(n)$, this appendix considers the abstract algebras $P_r(x)$ and $\mathbb{MP}_{r,k}(x)$ over $\mathbb{C}(x)$ for x an indeterminate. These results can all be applied to the centralizer algebra case by specializing x to an integer $n \geq 2r$.

In analogy with the construction of the diagram-like basis as a projection of the diagram basis of $P_r(n)$, we can define the *orbit-like basis* by projecting the orbit basis of $P_r(n)$:

$$\mathcal{O}_{\tilde{\pi}} = s_{\boldsymbol{a}} \mathcal{T}_{\pi} s_{\boldsymbol{b}}$$

where $\pi \in \Pi_{2r}$ is any set partition so that $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \tilde{\pi}$. For a multiset partition $\tilde{\pi}$, define $m_{\tilde{B}}(\tilde{\pi})$ to be the multiplicity of the block $\tilde{B}$ in $\tilde{\pi}$ and write

$$m(\tilde{\pi})! = \prod_{\substack{\tilde{B} \in \tilde{\pi} \\ \text{distinct}}} m_{\tilde{B}}(\tilde{\pi})$$

where the product is over *distinct* blocks of $\tilde{\pi}$. For a set S and a multiset $\tilde{B}$, write $\tilde{B}|_S$ for the multiset obtained by removing any elements not in S. For a multiset partition $\tilde{\pi} = \{\tilde{B}_1, \ldots, \tilde{B}_\ell\}$, write $\tilde{\pi}|_S = \{\tilde{B}_1|_S, \ldots, \tilde{B}_\ell|_S\}$.

Theorem A.1. *For $\tilde{\pi} \in \tilde{\Pi}_{2r,k}$ whose unbarred entries have multiplicity given by $\boldsymbol{a} \in W_{r,k}$, write*

$$\omega(\tilde{\pi}) = \frac{m(\tilde{\pi})!}{|\mathfrak{S}_{\boldsymbol{a}}|} \prod_{\tilde{B} \in \tilde{\pi}} m(\tilde{B}|_{[\bar{k}]})!.$$

Then the map

$$\begin{aligned} \varphi : \tilde{P}_{r,k}(x) &\to \mathbb{MP}_{r,k}(x) \\ \mathcal{O}_{\tilde{\pi}} &\mapsto \omega(\tilde{\pi})\mathcal{X}_{\tilde{\pi}} \end{aligned}$$

where $\{\mathcal{X}_{\tilde{\pi}} : \tilde{\pi} \in \tilde{\Pi}_{2r,k}\}$ is the orbit basis of Orellana and Zabrocki is an isomorphism of algebras.

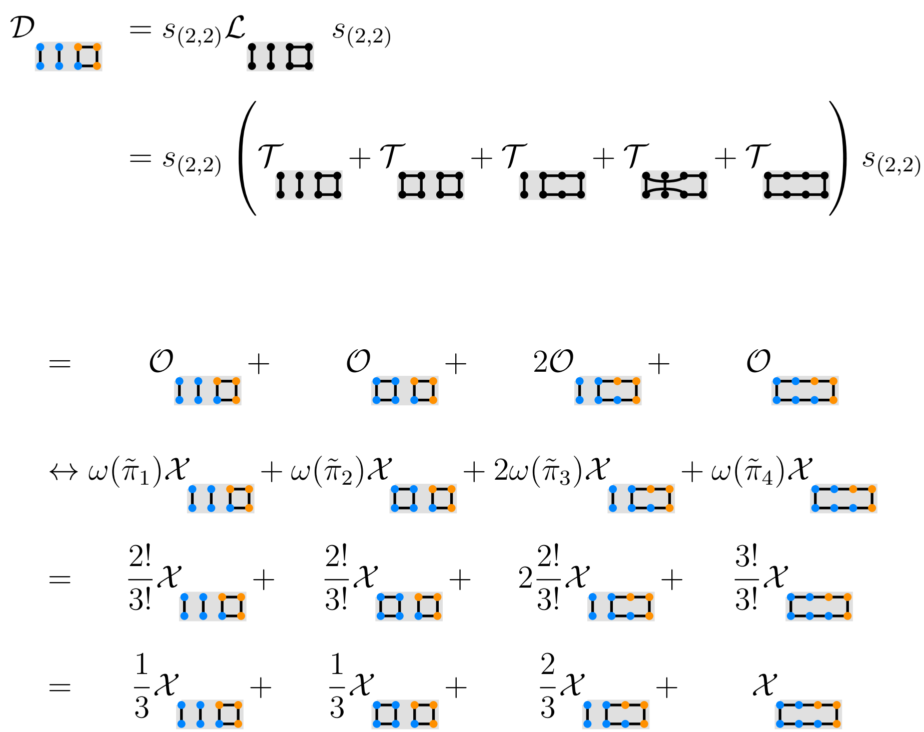
Because each $\omega(\tilde{\pi})$ is nonzero, it's clear that this map is an isomorphism of vector spaces–the remainder of this appendix is devoted to proving that it respects the multiplication. First, we observe that such an isomorphism gives us the following formula for the change of basis from Orellana and Zabrocki's orbit basis to the diagram-like basis:

$$\begin{aligned} \mathcal{D}_{\tilde{\pi}} &= \varphi(s_{\boldsymbol{a}} L_\pi s_{\boldsymbol{b}}) \\ &= \varphi(\sum_{\nu \leq \pi} s_{\boldsymbol{a}} \mathcal{T}_\nu s_{\boldsymbol{b}}) \\ &= \sum_{\tilde{\nu} \leq \tilde{\pi}} c_{\tilde{\nu},\tilde{\pi}} \varphi(\mathcal{O}_{\tilde{\pi}}) \end{aligned}$$

Where for a fixed π such that $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \tilde{\pi}$, $c_{\tilde{\nu},\tilde{\pi}}$ is the number of $\nu \leq \pi$ such that $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\nu) = \tilde{\nu}$

$$= \sum_{\tilde{\nu} \leq \tilde{\pi}} c_{\tilde{\nu},\tilde{\pi}} \omega(\tilde{\nu}) \mathcal{X}_{\tilde{\nu}}$$

Example A.2. We expand the following diagram-like basis element in the orbit basis of Orellana and Zabrocki:



$$\mathcal{D} = s_{(2,2)} \mathcal{L} \, s_{(2,2)}$$
$$= s_{(2,2)} \left(\mathcal{T} + \mathcal{T} + \mathcal{T} + \mathcal{T} + \mathcal{T} \right) s_{(2,2)}$$
$$= \mathcal{O} + \mathcal{O} + 2\mathcal{O} + \mathcal{O}$$
$$\leftrightarrow \omega(\tilde{\pi}_1)\mathcal{X} + \omega(\tilde{\pi}_2)\mathcal{X} + 2\omega(\tilde{\pi}_3)\mathcal{X} + \omega(\tilde{\pi}_4)\mathcal{X}$$
$$= \frac{2!}{3!}\mathcal{X} + \frac{2!}{3!}\mathcal{X} + 2\frac{2!}{3!}\mathcal{X} + \frac{3!}{3!}\mathcal{X}$$
$$= \frac{1}{3}\mathcal{X} + \frac{1}{3}\mathcal{X} + \frac{2}{3}\mathcal{X} + \mathcal{X}$$

Section A.1

Definitions and Enumerative Results

The product formula for the orbit basis $\{\mathcal{T}_\pi : \pi \in \Pi_{2r}\}$ for $P_r(n)$ use set partitions which include unbarred, barred, and double-barred elements. To that end, we make the following definitions. For $\pi, \nu \in \Pi_{2r}$, write Γ^π_ν for the set of set partitions γ of $[r] \cup [\overline{r}] \cup [\overline{\overline{r}}]$ such that $\gamma|_{[r]\cup[\overline{r}]} = \pi$ and $\gamma|_{[\overline{r}]\cup[\overline{\overline{r}}]} = \overline{\nu}$ where $\overline{\nu}$ is the result of adding a bar to each element in ν. For such a γ, write $\beta_\gamma = \{S \in \gamma : \forall i \in S, i \in [\overline{r}]\}$ for the set of blocks of γ contained entirely in the middle. Write

$$b_\gamma(x) = (x - \ell(\gamma|_{[r]\cup[\overline{\overline{r}}]}))_{\ell(\beta_\gamma)}$$

where $(a)_n = a(a-1)\cdots(a-n+1)$. The product formula for the orbit basis is

$$\mathcal{T}_\pi \mathcal{T}_\nu = \sum_{\gamma \in \Gamma^\pi_\nu} b_\gamma(x) \mathcal{T}_\gamma$$

where the γ in the subscript is understood to be an element of Π_{2r} by taking the restriction $\gamma|_{[r]\cup[\overline{\overline{r}}]}$ and removing a bar from each double-barred entry (see Theorem 4.14 of [4] for details).

For the orbit basis $\{\mathcal{X}_{\tilde{\pi}} : \tilde{\pi} \in \tilde{\Pi}_{2r,k}\}$ of $\mathbb{MP}_{r,k}(n)$ we make similar definitions. For $\tilde{\pi}, \tilde{\nu} \in \tilde{\Pi}_{2r,k}$, write $\tilde{\Gamma}^{\tilde{\pi}}_{\tilde{\nu}}$ for the set of multiset partitions $\tilde{\gamma}$ with r elements each from $[k]$, $[\overline{k}]$, and $[\overline{\overline{k}}]$ such that $\tilde{\gamma}|_{[k]\cup[\overline{k}]} = \tilde{\pi}$ and $\tilde{\gamma}|_{[\overline{k}]\cup[\overline{\overline{k}}]} = \overline{\tilde{\nu}}$. For such a $\tilde{\gamma}$, write $\beta_{\tilde{\gamma}} = \left\{\!\!\left\{\tilde{S} \in \tilde{\gamma} : \forall i \in \tilde{S}, i \in [\overline{k}]\right\}\!\!\right\}$. For any multiset partition $\tilde{\rho}$, write $m_{\tilde{S}}(\tilde{\rho})$ for the multiplicity of $\tilde{S}$ in $\tilde{\rho}$ and

$$m(\tilde{\rho})! = \prod_{\substack{\tilde{S}\in\tilde{\rho} \\ \text{distinct}}} m_{\tilde{S}}(\tilde{\rho})!$$

Finally, write

$$\tilde{b}_{\tilde{\gamma}}(x) = \frac{(x - \ell(\tilde{\gamma}|_{[k]\cup[\bar{\bar{k}}]}))_{\ell(\beta_{\tilde{\gamma}})}}{m(\beta_{\tilde{\gamma}})!}$$
$$a_{\tilde{\gamma}} = \prod_{\substack{\tilde{S}\in\tilde{\gamma}|_{[k]\cup[\bar{\bar{k}}]} \\ \text{distinct}}} \frac{\ell(\tilde{\gamma}_{\tilde{S}})!}{m(\tilde{\gamma}_{\tilde{S}})!}$$

where $\tilde{\gamma}_{\tilde{S}} = \left\{\!\!\left\{\tilde{T}\in\tilde{\gamma} : \tilde{T}|_{[k]\cup[\bar{\bar{k}}]} = \tilde{S}\right\}\!\!\right\}$. Then, the product for the orbit basis of $\mathbb{MP}_{r,k}(x)$ is given by

$$\mathcal{X}_{\tilde{\pi}}\mathcal{X}_{\tilde{\nu}} = \sum_{\tilde{\gamma}\in\tilde{\Gamma}^{\tilde{\pi}}_{\tilde{\nu}}} a_{\tilde{\gamma}}\tilde{b}_{\tilde{\gamma}}(x)\mathcal{X}_{\tilde{\gamma}}$$

where the $\tilde{\gamma}$ in the subscript is understood to be an element of $\tilde{P}i_{2r,k}$ by taking the restriction $\gamma|_{[k]\cup[\bar{\bar{k}}]}$ and removing a bar from each double-barred entry (see Section 3 of [41] for details).

To handle these set and multiset partitions on three alphabets combinatorially, we will want to extend the notation of our painting function $\kappa_{\boldsymbol{a},\boldsymbol{b}}$ to them. In particular, $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\gamma)$ will be the result of replacing the unbarred, barred, and double-barred elements according to $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ respectively. It will be useful later to write $\Gamma^{\pi}_{\nu}(\tilde{\mu})$ for the set of $\gamma\in\Gamma^{\pi}_{\nu}$ such that $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\gamma) = \tilde{\mu}$.

For $\pi,\nu\in\Pi_{2r}$ with $\pi|_{[\bar{r}]} = \overline{\nu}|_{\bar{r}}$, let $\pi * \nu \in \Gamma^{\pi}_{\nu}$ be the set partition obtained by placing the diagram of π atop the diagram of ν and identifying the corresponding vertices in the center. This set partition plays a central role because any $\gamma\in\Gamma^{\pi}_{\nu}$ only differs from $\pi * \nu$ by connecting some blocks on the very top and very bottom. That is, if we denote by $\mathrm{break}(\gamma)$ the result of splitting each block of γ which does not contain a vertex in the middle into it's restriction to $[r]$ and restriction to $[\overline{r}]$, then $\gamma \in \Gamma_\nu^\pi$ if and only if $\mathrm{break}(\gamma) = \pi * \nu$.

To analyze the orbit basis of $\mathbb{MP}_{r,k}(n)$, we want to investigate how $\pi * \nu$ acts when ν is acted upon by some permutation. This inspires the definition of a number of subgroups of permutations. For ρ a set partition of $[r]$ and $\boldsymbol{a} \in W_{r,k}$, define

$$\mathfrak{S}_{\boldsymbol{a}}^\rho = \{\sigma \in \mathfrak{S}_{\boldsymbol{a}} : \sigma.\rho = \rho\}$$

where the action $\sigma.\rho$ applies σ to each element of each block of ρ. This subgroup factors as a semidirect product

$$S_{\boldsymbol{a}}^\rho = X_{\boldsymbol{a}}^\rho \ltimes Y_{\boldsymbol{a}}^\rho$$

where the permutations in $X_{\boldsymbol{a}}^\rho$ permute whole blocks and the permutations in $Y_{\boldsymbol{a}}^\rho$ permute only within blocks of ρ.

Given $\pi \in \Pi_{2r}$, consider the subgroup $A_{\boldsymbol{a},\boldsymbol{b}}^\pi$ of $X_{\boldsymbol{b}}^{\pi|_{[\overline{r}]}}$ which only permutes blocks in $\pi|_{[\overline{r}]}$ if they are part of blocks of π that are painted identically by $\kappa_{\boldsymbol{a},\boldsymbol{b}}$. Let $B_{\boldsymbol{a},\boldsymbol{b}}^\nu$ be the analogous subgroup for the restrictions to the top. More precisely,

$$A_{\boldsymbol{a},\boldsymbol{b}}^\pi = \{x \in X_{\boldsymbol{b}}^{\pi|_{[\overline{r}]}} : \forall S, T \in \pi, S|_{[\overline{r}]} = x(T|_{[\overline{r}]}) \implies \kappa_{\boldsymbol{a},\boldsymbol{b}}(S) = \kappa_{\boldsymbol{a},\boldsymbol{b}}(T)\}$$
$$B_{\boldsymbol{b},\boldsymbol{c}}^\nu = \{x \in X_{\boldsymbol{b}}^{\nu|_{[r]}} : \forall S, T \in \nu, S|_{[r]} = x(T|_{[r]}) \implies \kappa_{\boldsymbol{b},\boldsymbol{c}}(S) = \kappa_{\boldsymbol{b},\boldsymbol{c}}(T)\}$$

Put another way, these permutations σ of the blocks in the middle of $\pi * \nu$ do not change the resulting multiset partition. That is, $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \sigma.\nu)$ for $\sigma \in A^{\pi}_{\boldsymbol{a},\boldsymbol{b}}$ or $\sigma \in B^{\nu}_{\boldsymbol{b},\boldsymbol{c}}$.

Finally, we collect up formulas for the sizes of these subgroups. To that end, it will be useful to consider the following multiset partitions obtained by restricting to particular blocks.

$$\begin{aligned}
\tilde{\pi}_+ &= \{\tilde{S} \in \tilde{\pi} : \forall i \in S, i \in [k]\} \\
\tilde{\pi}_- &= \{\tilde{S} \in \tilde{\pi} : \forall i \in S, i \in [\overline{k}]\} \\
\tilde{\gamma}_\pm &= \{\tilde{S} \in \tilde{\pi} : \forall i \in S, i \in [k] \cup [\overline{\overline{k}}]\}
\end{aligned}$$

We think of $\tilde{\pi}_+$ (resp. $\tilde{\pi}_-$) as the blocks contained entirely in the top (resp. bottom) of $\tilde{\pi}$ and $\tilde{\gamma}_\pm$ as the blocks of $\tilde{\gamma}$ which have no vertex in the middle row.

Lemma A.3. *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in W_{r,k}$, $\pi, \nu \in \Pi_{2r}$, $\tilde{\pi} = \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi), \tilde{\nu} = \kappa_{\boldsymbol{b},\boldsymbol{c}}(\nu)$ and $\tilde{\gamma} \in \tilde{\Gamma}^{\tilde{\pi}}_{\tilde{\nu}}$. Then,*

$$\begin{aligned}
|X^{\rho}_{\boldsymbol{a}}| &= m(\kappa_{\boldsymbol{a}}(\rho))! \\
|Y^{\rho}_{\boldsymbol{a}}| &= \prod_{\tilde{B} \in \tilde{\pi}} m(\tilde{B}|_{[\overline{k}]})! \\
\left|A^{\pi}_{\boldsymbol{a},\boldsymbol{b}}\right| &= \frac{m(\tilde{\pi})!}{m(\tilde{\pi}_+)!} \\
\left|B^{\nu}_{\boldsymbol{b},\boldsymbol{c}}\right| &= \frac{m(\tilde{\nu})!}{m(\tilde{\nu}_-)!} \\
\left|A^{\pi}_{\boldsymbol{a},\boldsymbol{b}} \cap B^{\nu}_{\boldsymbol{b},\boldsymbol{c}}\right| &= \frac{m(\tilde{\gamma})!}{m(\tilde{\gamma}_\pm)!}
\end{aligned}$$

Proof. The first two equalities are clear. The next two follow from the observation that the only blocks of $\tilde{\pi}$ that contribute to $A^{\pi}_{\boldsymbol{a},\boldsymbol{b}}$ are the ones which touch the bottom, so we cancel out the contribution of those contained entirely in the top.

For the last equality, we can think of $A^{\pi}_{\boldsymbol{a},\boldsymbol{b}} \cap B^{\nu}_{\boldsymbol{b},\boldsymbol{c}}$ as the permutations of the middle of $\pi * \nu$ which only permute blocks which are the restrictions of blocks painted the same in $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu)$. The number of such permutations is $\frac{m(\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi*\nu))!}{m(\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi*\nu)_{\pm})!}$. Because $\tilde{\gamma}$ only differs from $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu)$ by blocks which don't touch the center (whose contributions are all canceled) we can make the substitution of $\tilde{\gamma}$ for $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu)$. $\square$

Section A.2

Proof of the Isomorphism

Proof of Theorem A.1. Let $\tilde{\pi}, \tilde{\nu} \in \tilde{\Pi}_{2r}$ and let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{b}', \boldsymbol{c} \in W_{r,k}$ be such that there exist $\pi, \nu \in \Pi_{2r}$ so that $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi) = \tilde{\pi}$ and $\kappa_{\boldsymbol{b}',\boldsymbol{c}}(\nu) = \tilde{\nu}$. Note that when $\boldsymbol{b} \neq \boldsymbol{b}'$, we have $\mathcal{O}_{\tilde{\pi}}\mathcal{O}_{\tilde{\nu}} = 0$ and $\mathcal{X}_{\tilde{\pi}}\mathcal{X}_{\tilde{\nu}} = 0$, so we need only address the case when $\boldsymbol{b} = \boldsymbol{b}'$:

$$\begin{aligned}
\mathcal{O}_{\tilde{\pi}}\mathcal{O}_{\tilde{\nu}} &= \frac{1}{|\mathfrak{S}_{\boldsymbol{b}}|} \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} s_{\boldsymbol{a}} \mathcal{T}_{\pi} \mathcal{T}_{\sigma.\nu} s_{\boldsymbol{c}} \\
&= \frac{1}{|\mathfrak{S}_{\boldsymbol{b}}|} \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} \sum_{\gamma \in \Gamma^{\pi}_{\sigma.\nu}} b_{\gamma}(x) s_{\boldsymbol{a}} \mathcal{T}_{\gamma} s_{\boldsymbol{b}}
\end{aligned}$$

To simplify notation, we will write $\tilde{\gamma} = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\gamma)$. Note that $b_{\gamma}(x) = \tilde{b}_{\tilde{\gamma}}(x) m(\beta_{\tilde{\gamma}})!$, so we can rewrite the expression as follows.

$$= \frac{1}{|\mathfrak{S}_{\boldsymbol{b}}|} \sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} \sum_{\gamma \in \Gamma^{\pi}_{\sigma.\nu}} \tilde{b}_{\tilde{\gamma}}(x) m(\beta_{\tilde{\gamma}})! \mathcal{O}_{\tilde{\gamma}}$$

We then partition the sum over the possible multiset partitions $\tilde{\mu}$ that could arise from γ in this sum, noting that $\kappa_{\boldsymbol{b},\boldsymbol{c}}(\sigma.\nu) = \tilde{\nu}$, and then we swap the order of summation

$$= \frac{1}{|\mathfrak{S}_{\boldsymbol{b}}|} \sum_{\tilde{\mu} \in \tilde{\Gamma}^{\tilde{\pi}}_{\tilde{\nu}}} \tilde{b}_{\tilde{\mu}}(x) m(\beta_{\tilde{\mu}})! \left(\sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} \sum_{\substack{\gamma \in \Gamma^{\pi}_{\sigma.\nu} \\ \tilde{\gamma} = \tilde{\mu}}} 1 \right) \mathcal{O}_{\tilde{\mu}}$$

Applying φ to both sides, we see that the coefficient of $\mathcal{X}_{\tilde{\tau}}$ in $\varphi(\mathcal{O}_{\tilde{\pi}}\mathcal{O}_{\tilde{\nu}})$ is:

$$\varphi(\mathcal{O}_{\tilde{\pi}}\mathcal{O}_{\tilde{\nu}})|_{\mathcal{X}_{\tilde{\tau}}} = \frac{\omega(\tilde{\tau})}{|\mathfrak{S}_{\boldsymbol{b}}|} \sum_{\substack{\tilde{\mu} \in \Gamma^{\tilde{\pi}}_{\tilde{\nu}} \\ \tilde{\mu}|_{[k] \cup [\bar{\bar{k}}]} = \tilde{\tau}}} \tilde{b}_{\tilde{\mu}}(x) m(\beta_{\tilde{\mu}})! \left(\sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} |\Gamma^{\pi}_{\sigma.\nu}(\tilde{\mu})| \right)$$

We now compare this to the same coefficient in $\varphi(\mathcal{O}_{\tilde{\pi}})\varphi(\mathcal{O}_{\tilde{\nu}})$.

$$\varphi(\mathcal{O}_{\tilde{\pi}})\varphi(\mathcal{O}_{\tilde{\nu}})|_{\mathcal{X}_{\tilde{\tau}}} = \omega(\tilde{\pi})\omega(\tilde{\nu}) \sum_{\substack{\tilde{\mu} \in \tilde{\Gamma}^{\tilde{\pi}}_{\tilde{\nu}} \\ \tilde{\mu}|_{[k] \cup [\bar{\bar{k}}] = \tilde{\tau}}}} a_{\tilde{\mu}} \tilde{b}_{\tilde{\mu}}(x)$$

The goal is now to show that these two rather unpleasant quantities are the same. We will leverage their similarities—namely that they sum over the same objects and each include a factor of $\tilde{b}_{\tilde{\mu}}(x)$ in each term—to simplify the task. It would suffice to show the following equality:

$$\frac{\omega(\tilde{\tau})}{|\mathfrak{S}_{\boldsymbol{b}}|} m(\beta_{\tilde{\mu}})! \left(\sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} |\Gamma^{\pi}_{\sigma.\nu}(\tilde{\mu})| \right) = \omega(\tilde{\pi})\omega(\tilde{\nu}) a_{\tilde{\mu}}$$

By using the definition of $\omega(\tilde{\pi})$ and noticing that $\tilde{\tau}|_{[\bar{\bar{k}}]} = \tilde{\nu}|_{[\bar{\bar{k}}]}$, we are able to rearrange the above equality to the following:

$$\sum_{\sigma \in \mathfrak{S}_{\boldsymbol{b}}} |\Gamma^{\pi}_{\sigma.\nu}(\tilde{\mu})| = a_{\tilde{\mu}} \frac{m(\tilde{\pi})!m(\tilde{\nu})!}{m(\tilde{\tau})!m(\beta_{\tilde{\mu}})!} \prod_{\tilde{B} \in \tilde{\pi}} m(\tilde{B}|_{[\overline{k}]})!$$

Then using the formulas in Lemma A.3 for the sizes of the subgroups, we get:

$$\begin{aligned} &= a_{\tilde{\mu}} \frac{m(\tilde{\pi}_+)!m(\tilde{\nu}_-)!}{m(\tilde{\tau})!m(\beta_{\tilde{\mu}})!} \frac{m(\tilde{\mu})!}{m(\tilde{\mu}_{\pm})!} \frac{\left|A^{\pi}_{\boldsymbol{a},\boldsymbol{b}}\right| \left|B^{\nu}_{\boldsymbol{b},\boldsymbol{c}}\right| \left|Y^{\pi|_{[\overline{r}]}}_{\boldsymbol{b}}\right|}{\left|A^{\pi}_{\boldsymbol{a},\boldsymbol{b}} \cap B^{\nu}_{\boldsymbol{b},\boldsymbol{c}}\right|} \\ &= a_{\tilde{\mu}} \frac{m(\tilde{\pi}_+)!m(\tilde{\nu}_-)!}{m(\tilde{\tau})!m(\beta_{\tilde{\mu}})!} \frac{m(\tilde{\mu})!}{m(\tilde{\mu}_{\pm})!} \left|A^{\pi}_{\boldsymbol{a},\boldsymbol{b}} B^{\nu}_{\boldsymbol{b},\boldsymbol{c}} Y^{\pi|_{[\overline{r}]}}_{\boldsymbol{b}}\right| \end{aligned}$$

And finally, using the fact that $\tilde{\tau} = \tilde{\mu}|_{[k]\cup[\overline{\overline{k}}]}$, and so for a block $\tilde{S} \in \tilde{\mu}|_{[k]\cup[\overline{\overline{k}}]}$ we have that $\ell(\tilde{\mu}_{\tilde{S}}) = m_{\tilde{S}}(\tilde{\tau})$:

$$\begin{aligned} &= \left(\frac{m(\tilde{\mu})!}{m(\beta_{\tilde{\mu}})!} \prod_{\substack{\tilde{S} \in \tilde{\mu}|_{[k]\cup[\overline{\overline{k}}]} \\ \text{distinct}}} \frac{1}{m(\tilde{\mu}_{\tilde{S}})!} \right) \frac{m(\tilde{\pi}_+)!m(\tilde{\nu}_-)!}{m(\tilde{\mu}_{\pm})!} \left|A^{\pi}_{\boldsymbol{a},\boldsymbol{b}} B^{\nu}_{\boldsymbol{b},\boldsymbol{c}} Y^{\pi|_{[\overline{r}]}}_{\boldsymbol{b}}\right| \\ &= \frac{m_+(\tilde{\pi})m_-(\tilde{\nu})}{m_{\pm}(\tilde{\mu})} \left|A^{\pi}_{\boldsymbol{a},\boldsymbol{b}} B^{\nu}_{\boldsymbol{b},\boldsymbol{c}} Y^{\pi|_{[\overline{r}]}}_{\boldsymbol{b}}\right| \end{aligned}$$

The proof of this equality will be carried out in two lemmas. First, Lemma A.6 will show that the set of σ such that $|\Gamma^{\pi}_{\sigma.\nu}(\tilde{\mu})| \neq 0$ is given by a translation of the product $A^{\pi}_{\boldsymbol{a},\boldsymbol{b}} B^{\nu'}_{\boldsymbol{b},\boldsymbol{c}} Y^{\pi|_{[\overline{r}]}}_{\boldsymbol{b}}$ where $\kappa_{\boldsymbol{b},\boldsymbol{c}}(\nu') = \kappa_{\boldsymbol{b},\boldsymbol{c}}(\nu)$ so $B^{\nu'}_{\boldsymbol{b},\boldsymbol{c}} \cong B^{\nu}_{\boldsymbol{b},\boldsymbol{c}}$. Finally, Lemma A.8 will show that for each such σ,

$$|\Gamma^{\pi}_{\sigma.\nu}(\tilde{\mu})| = \frac{m_+(\tilde{\pi})m_-(\tilde{\nu})}{m_{\pm}(\tilde{\mu})}.$$

Because this quantity is independent of σ, the value of the sum is simply the product of the two quantities. □

To set the stage for the final two lemmas, we first need to investigate a particular class of permutations in $\sigma \in \mathfrak{S}_{\boldsymbol{b}}^{\rho}$.

Lemma A.4. *Let $\pi, \nu \in \Pi_{2r}$ such that $\pi|_{[\overline{r}]} = \overline{\nu}|_{[\overline{r}]} = \rho$ and fix $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in W_{r,k}$. Then*

$$\{\sigma \in \mathfrak{S}_{\boldsymbol{b}}^{\rho} : \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \sigma.\nu) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu)\} = A_{\boldsymbol{a},\boldsymbol{b}}^{\pi} B_{\boldsymbol{b},\boldsymbol{c}}^{\nu} Y_{\boldsymbol{b}}^{\rho}.$$

Proof. First, note that σ factors as $\sigma = xy$ for $x \in X_{\boldsymbol{m}}^{\rho}$ and $y \in Y_{\boldsymbol{m}}^{\rho}$. Because $y.\nu = \nu$ for all ν, we need only determine which x can be factored into a product of an element of $A_{\boldsymbol{a},\boldsymbol{b}}^{\pi}$ and an element of $B_{\boldsymbol{b},\boldsymbol{c}}^{\nu}$.

One containment is straightforward. Suppose $x = x_1 x_2$ with $x_1 \in A_{\boldsymbol{a},\boldsymbol{b}}^{\pi}$ and $x_2 \in B_{\boldsymbol{b},\boldsymbol{c}}^{\nu}$. Consider a block in $\pi * \nu$. Although the bottom half of this block may be different in $\pi * x_2.\nu$, the condition that $x_2 \in B_{\boldsymbol{b},\boldsymbol{c}}^{\nu}$ guarantees that it is not different in $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * x_2.\nu)$. Hence, $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * x_2.\nu) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu)$ for any $\pi \in \Pi_{2r}$ with $\pi|_{[\overline{r}]} = \overline{\nu}|_{[\overline{r}]}$. Analogously, we see that $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi.{x_1}^{-1} * \nu) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu)$ for any $\nu \in \Pi_{2r}$ with $\pi|_{[\overline{r}]} = \overline{\nu}|_{[\overline{r}]}$. Hence,

$$\begin{aligned} \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * x_1 x_2.\nu) &= \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi.{x_1}^{-1} * x_2.\nu) \\ &= \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi.{x_1}^{-1} * \nu) \\ &= \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu) \end{aligned}$$

For the other containment, suppose that $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * x.\nu) = \tilde{\mu}$. We use the following convention for indexing the blocks of $\pi * \nu$. Write $\rho = \{M_1 < \cdots < M_\ell\}$ for the restrictions of the blocks of $\pi * \nu$ to $[\overline{r}]$ in last-letter order, and write S_i for the block of $\pi * \nu$ with $M_i \subseteq S_i$. An element $\sigma \in X^{\rho}_{\boldsymbol{m}}$ permutes the blocks of ρ and hence the indices $[\ell]$. For $S_i \in \pi * \nu$, it will be helpful to write $\sigma(S_i)$ for the block in $\pi * \sigma.\nu$ such that $S_i|_{[r]\cup[\overline{r}]} = \sigma(S_i)|_{[r]\cup[\overline{r}]}$. Equivalently, $\sigma(S_i)$ is obtained by replacing the bottom row $S_i|_{[\overline{\overline{r}}]}$ with $S_{\sigma^{-1}(i)}|_{[\overline{\overline{r}}]}$.

For a fixed $\sigma \in X^{\rho}_{\boldsymbol{b}}$ and $\tilde{S}$ a multiset from $[k] \cup [\overline{k}] \cup [\overline{\overline{k}}]$, write

$$W_{\sigma,\tilde{S}} = \{i : \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\sigma(S_i)) = \tilde{S}\}.$$

For an example of these sets, see Example A.5. The following two properties can be observed in this example—we show that they hold in general.

1. For a fixed $\tilde{R} \in \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi)$ such that $\tilde{R} \cap [\overline{k}] \neq \emptyset$,

$$\bigsqcup_{\substack{\tilde{S}\in\tilde{\mu} \\ \tilde{S}|_{[k]\cup[\overline{k}]}=\tilde{R}}} W_{1,\tilde{S}} = \bigsqcup_{\substack{\tilde{S}\in\tilde{\mu} \\ \tilde{S}|_{[k]\cup[\overline{k}]}=\tilde{R}}} W_{x,\tilde{S}}$$

   This follows just about immediately from the fact that $\sigma(S_i)|_{[r]\cup[\overline{r}]} = S_i|_{[r]\cup[\overline{r}]}$ for any $\sigma \in \mathfrak{S}^{\rho}_{\boldsymbol{b}}$.

$$\begin{aligned}
\bigsqcup_{\substack{\tilde{S}\in\tilde{\mu} \\ \tilde{S}|_{[k]\cup[\overline{k}]}=\tilde{R}}} W_{x,\tilde{S}} &= \left\{i : \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(x(S_i))|_{[k]\cup[\overline{k}]} = \tilde{R}\right\} \\
&= \left\{i : \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(S_i)|_{[k]\cup[\overline{k}]} = \tilde{R}\right\} \\
&= \bigsqcup_{\substack{\tilde{S}\in\tilde{\mu} \\ \tilde{S}|_{[k]\cup[\overline{k}]}=\tilde{R}}} W_{1,\tilde{S}}
\end{aligned}$$

Note that the assumption that $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * x.\nu) = \tilde{\mu}$ is necessary here so that the unions on either side of the equality are over the same set of $\tilde{S}$.

2. For a fixed $\tilde{S} \in \tilde{\mu}$ with $\tilde{S} \cap [\overline{k}] \neq \emptyset$,

$$|W_{x,\tilde{S}}| = |W_{1,\tilde{S}}|$$

For $\sigma \in \mathfrak{S}_{\boldsymbol{b}}^{\rho}$ and $\tilde{S} \in \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \sigma.\nu)$ with $\tilde{S} \cap [\overline{k}] \neq \emptyset$, we have

$$\begin{aligned} \left|W_{\sigma,\tilde{S}}\right| &= \left|\left\{i : \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\sigma(S_i)) = \tilde{S}\right\}\right| \\ &= m_{\tilde{S}}(\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \sigma.\nu)) \end{aligned}$$

The statement then follows from the assumption that $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * x.\nu) = \tilde{\mu}$.

These two facts allow us to construct a permutation in the following way. Fixing $\tilde{R} \in \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi)$ such that $\tilde{R} \cap [\overline{k}] \neq \emptyset$, there exists a permutation $h_{\tilde{R}}$ of $\{i : \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(S_i)|_{[k]\cup[\overline{k}]} = \tilde{R}\}$ such that $h_{\tilde{R}}(W_{x,\tilde{S}}) = W_{1,\tilde{S}}$ for all $\tilde{S}$ with $\tilde{S}|_{[k]\cup[\overline{k}]} = \tilde{R}$. Because $h_{\tilde{R}}$ by definition permutes only blocks which restrict to the same block in $\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi)$, we see $h_{\tilde{R}}$ is an element of $A^{\pi}_{\boldsymbol{a},\boldsymbol{b}}$. Now define $h \in A^{\pi}_{\boldsymbol{a},\boldsymbol{b}}$ by

$$h := \prod_{\substack{\tilde{R} \in \tilde{\pi}^{\boldsymbol{t},\boldsymbol{m}} \\ \tilde{R} \cap [\overline{k}] \neq \emptyset}} h_{\tilde{R}}$$

It remains only to show that $hx \in B^{\nu}_{\boldsymbol{m},\boldsymbol{b}}$. Fix $i \in [\ell]$ and let $\tilde{S} = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(hx(S_i))$.

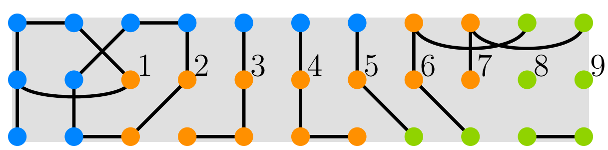
$$\begin{aligned}\tilde{S} &= \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(hx(S_i))\\ &= \kappa_{\boldsymbol{a},\boldsymbol{b}}(S_i|_{[r]\cup[\overline{r}]}) \cup \kappa_{\boldsymbol{c}}(S_{(hx)^{-1}(i)})\end{aligned}$$

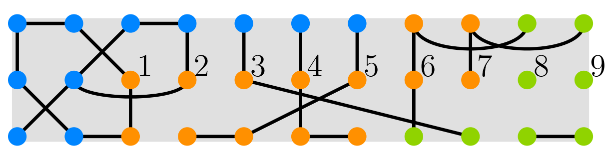
Then by the fact that $h \in A^{\pi}_{\boldsymbol{a},\boldsymbol{b}}$,

$$\begin{aligned} &= \kappa_{\boldsymbol{a},\boldsymbol{b}}(S_{h^{-1}(i)}|_{[r]\cup[\overline{r}]}) \cup \kappa_{\boldsymbol{c}}(S_{x^{-1}(h^{-1}(i))})\\ &= \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(x(S_{h^{-1}(i)}))\end{aligned}$$

Then $h^{-1}(i) \in W_{x,\tilde{S}}$, so $i \in h(W_{x,\tilde{S}}) = W_{1,\tilde{S}}$. Hence, $\tilde{S}_i = \tilde{S} = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(hx(S_i))$ for each i and so $hx \in B^{\pi}_{\boldsymbol{t},\boldsymbol{m}}$, meaning $x \in A^{\pi}_{\boldsymbol{t},\boldsymbol{m}} B^{\nu}_{\boldsymbol{m},\boldsymbol{b}}$. Thus $\sigma = xy \in A^{\pi}_{\boldsymbol{t},\boldsymbol{m}} B^{\nu}_{\boldsymbol{m},\boldsymbol{b}} Y^{\rho}_{\boldsymbol{m}}$. $\square$

Example A.5. We label the blocks of ρ in the middle of the diagram $\pi * \nu$ in last-letter order and let $x = (1\,2)(3\,5\,6)$. We then apply these labels to our diagrams of $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu)$ and $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * x.\nu)$.

$\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu) =$ 

$\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * x.\nu) =$ 

Now we can read off that $x(S_2) = \{3, 4, \overline{2}, \overline{4}, \overline{\overline{1}}\}$ and $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(x(S_2)) = \left\{\!\!\left\{1, 1, \overline{1}, \overline{2}, \overline{\overline{1}}\right\}\!\!\right\}$ by simply looking at the blocks labeled 2 in the diagrams.

Consider $\tilde{R} = \{\{1, \overline{2}\}\} \in \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi)$. There are two distinct blocks in $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu)$ which restrict to this $\tilde{R}$: $\tilde{S} = \left\{\left\{1, \overline{2}, \overline{\overline{2}}, \overline{\overline{2}}\right\}\right\}$ and $\tilde{S}' = \left\{\left\{1, \overline{2}, \overline{\overline{3}}\right\}\right\}$.

$$W_{1,\tilde{S}} = \{3, 4\} \qquad W_{1,\tilde{S}'} = \{5\}$$

$$W_{x,\tilde{S}} = \{4, 5\} \qquad W_{x,\tilde{S}'} = \{3\}$$

Notice that the sets in each column have the same size and the union across rows is always $\{3, 4, 5\}$.

Lemma A.6. *Fix $\pi, \nu \in \Pi_{2r}$ such that $\pi|_{[\overline{r}]} = \overline{\nu}_{[\overline{r}]} = \rho$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in W_{r,k}$. Let $\tilde{\mu} \in \tilde{\Gamma}^{\kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi)}_{\kappa_{\boldsymbol{b},\boldsymbol{c}}(\nu)}$. The set of $\sigma \in S_{\boldsymbol{m}}$ for which there exists a $\gamma \in \Gamma^{\pi}_{\sigma.\nu}(\tilde{\mu})$ is given by*

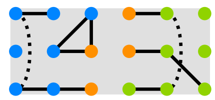
$$A^{\pi}_{\boldsymbol{a},\boldsymbol{b}} B^{\sigma_0.\nu}_{\boldsymbol{b},\boldsymbol{c}} Y^{\rho}_{\boldsymbol{b}} \sigma_0$$

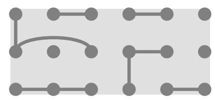
for some $\sigma_0 \in S_{\boldsymbol{b}}$.

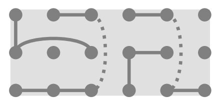
Proof. Let $\pi, \nu \in \Pi_{2r}$. If there exists $\gamma \in \Gamma^{\pi}_{\nu}(\tilde{\mu})$, then

$$\mathrm{break}(\tilde{\mu}) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu).$$

Conversely if $\mathrm{break}(\tilde{\mu}) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu)$ we can construct a $\gamma \in \Gamma^{\pi}_{\nu}(\tilde{\mu})$ as follows. For each block of $\tilde{\mu}$ broken into $\tilde{T}$ in the top and $\tilde{B}$ in the bottom, find blocks T and B in π and ν for which $\kappa_{\boldsymbol{a},\boldsymbol{b}}(T) = \tilde{T}$ and $\kappa_{\boldsymbol{b},\boldsymbol{c}}(B) = \tilde{B}$ and connect these blocks in $\pi * \nu$. After connecting such a pair for each block broken in μ, we have constructed the desired γ.

$\tilde{\mu} =$ 

$\pi * \nu =$ 

$\gamma =$ 

Hence, we are looking for the set of $\sigma \in \mathfrak{S}_{\boldsymbol{b}}$ for which $\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \sigma.\nu) = \text{break}(\tilde{\mu})$. Note that $\pi * \sigma.\nu$ only makes sense when $\sigma.\nu|_{[r]} = \rho$, so we need only consider $\sigma \in \mathfrak{S}_{\boldsymbol{b}}^{\rho}$.

Choose σ_0 so that $\text{break}(\tilde{\mu}) = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \sigma_0.\nu)$ and write $\nu' = \sigma_0.\nu$. Then the desired set of σ is precisely the permutations $\sigma \in S_{\boldsymbol{m}}^{\rho}$ such that

$$\kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * (\sigma\sigma_0^{-1}).\nu') = \kappa_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}(\pi * \nu')$$

Lemma A.4 tells us that this set is precisely those when

$$\sigma{\sigma_0}^{-1} \in A_{\boldsymbol{t},\boldsymbol{m}}^{\pi} B_{\boldsymbol{m},\boldsymbol{b}}^{\nu'} Y_{\boldsymbol{m}}^{\rho}$$

as desired.

$\square$

Lemma A.7. *Let μ be a set partition of $[m]$ and $\boldsymbol{a} \in W_{m,\ell}$ such that the blocks of $\kappa_{\boldsymbol{a}}(\mu) = \tilde{\mu}$ are all sets. Then the number of set partitions γ of $[m]$ such that $\kappa_{\boldsymbol{a}}(\gamma) = \tilde{\mu}$*

is

$$\frac{\boldsymbol{a}_1! \dots \boldsymbol{a}_\ell!}{m(\tilde{\mu})!}$$

Proof. First, observe that

$$\{\gamma : \kappa_{\boldsymbol{a}}(\gamma) = \tilde{\mu}\} = \mathfrak{S}_{\boldsymbol{a}}.\mu.$$

Because the blocks of $\tilde{\mu}$ are sets, no permutation of $\mathfrak{S}_{\boldsymbol{a}}$ swaps elements within a block of μ. Hence, the permutations which fix μ are precisely the ones that swap whole blocks, and the result is obtained by the orbit-stabilizer formula. ☐

Lemma A.8. *Fix* $\pi, \nu \in \Pi_{2r}$ *and suppose* $\tilde{\mu} \in \tilde{\Gamma}^{\kappa_{\boldsymbol{a},\boldsymbol{b}}(\tilde{\pi})}_{\kappa_{\boldsymbol{b},\boldsymbol{c}}(\tilde{\nu})}$. *If* $\Gamma^\pi_\nu(\tilde{\mu}) \neq \emptyset$, *then*

$$|\Gamma^\pi_\nu(\tilde{\mu})| = \frac{m(\tilde{\pi}_+)!m(\tilde{\nu}_-)!}{m(\tilde{\mu}_\pm)!}.$$

Proof. Let $\gamma \in \Gamma^\pi_\nu(\tilde{\mu})$. Because γ differs from $\pi * \nu$ by connecting some number of blocks in the very top and very bottom, we can recover γ uniquely from the partial matching of blocks of $(\pi * \nu)_\pm$ induced by $\gamma_\pm$. The question then becomes how many set partitions ρ on the blocks of $(\pi * \nu)_\pm$ there are such that $\kappa_{\boldsymbol{a},\boldsymbol{c}}(\rho) = \tilde{\mu}_\pm$. Because we are only connecting blocks on top to blocks on bottom, we can apply Lemma A.7 where the ℓ colors are the different multisets which appear in $\tilde{\mu}_\pm$. The number is then

$$\frac{m(\tilde{\mu}_\pm|_{[k]})!m(\tilde{\mu}_\pm|_{\overline{[k]}})!}{m(\tilde{\mu}_\pm)!} = \frac{m(\tilde{\pi}_+)!m(\nu_-)!}{m(\tilde{\mu}_\pm)!}$$

where $\tilde{\pi} = \kappa_{\boldsymbol{a},\boldsymbol{b}}(\pi)$ and $\tilde{\nu} = \kappa_{\boldsymbol{b},\boldsymbol{c}}(\nu)$. ☐

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